THE STATISTICAL BEHAVIOUR OF THE AVALANCHING PHOTODIODE (*)

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We have solved the general statistical problem for a uniform avalanche region, getting the mean value and the autocorrelation function for the output current pulse due to the injection, at a generical point x of the avalanche region, of an electron or a hole or an electron-hole-pair. Thereafter, as particular cases, we have obtained, as new results, the frequency response and the noise power spectrum for pair injection, showing that for moderate or large gain the noise power spectrum is proportional to the square of the mean response. The common cutoff frequency for injection at $x = 0$ is approximately given by:

$$f_T \simeq \frac{v^*}{2\pi L} a \quad \text{where } a = \frac{6}{M(0)} \left(\frac{\alpha}{\beta}\right)^{\frac{\lg M(0)}{1 - \lg M(0)}} \qquad \text{for } a \leq 4.5$$

$$f_T \simeq \frac{v^*}{2\pi L} 4.5 \qquad \text{for } a \geq 4.5$$

where L the length of the avalance region, $v^* = v_e v_h/(v_e + v_h)$ an effective velocity, with v_e and v_h the electron and hole drift velocity respectively, α and β the electron and hole ionization rates, $M(0)$ the multiplication factor for injection at point $x = 0$. As a consequence, in the time domain, the current response of the avalanche diode is fairly standard in shape: it is only its area the fluctuating quantity.

We have also calculated the single carrier average response, in the time domain, for different gains $M(0)$ and different ionization rate ratios β/α.

INTRODUCTION.

The need of a wide-band photodetector has stimulated in the last years the research in the field of avalanching photodiodes [1]. A simple photodiode connected to a small load to obtain a large bandwidth gives a small signal-to-noise ratio, limited by the thermal noise of the resistance. A better signal-to-noise-ratio is obtained with an avalanching photodiode because of internal amplification, and also a large bandwidth can be preserved.

Moreover, diodes operating in the avalanche zone has been proposed and utilized as detectors of nuclear radiation [2]. In the last year several papers have dealt with the problem of frequency response and noise properties of avalanching photodiodes [3, 4, 5, 6, 7], also in the case of unequal electron and hole ionization rates α and β and of unequal electron and hole saturated velocities v_e and v_h. However, the power spectral density of the noise in the avalanching photodiode has not been till now calculated exactly as a function of frequency.

We have completely characterized the general statistical problem of the output current of a uniform avalanching region in which the ionization rates α and β are constant, though, in general, unequal, and in which the carrier velocities v_e and v_h are assumed saturated, that is, constant. We have derived, with those assumptions, the mean response of the avalanching diode (i.e. the frequency response) and also its statistical properties (i.e. the noise power spectrum as a function of frequency).

Actually we have obtained the Laplace transforms $i_e(p|x)$ and $i(p|x)$ of the mean output current pulse for an electron and an electron-hole pair injected at a point x of the avalanche region and, besides it, the corresponding autocorrelation functions, so that the statistical properties of the random function output current has been characterized at the second order.

Finally, it is shown that the shape of the output current pulse for a given point of injection x is fairly standard, for charge gains $M(x) \geq 10$, being mainly the pulse area the fluctuating variable of this random function.

THE AVALANCHING DIODE STATISTICAL PROBLEM.

We consider a spatial region of length L for the avalanche zone (fig. 1) and assume that electrons move with constant drift velocity v_e in the x-axis direction while the holes move in the opposite direction with constant

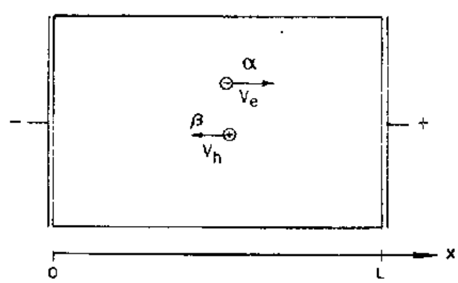

Fig. 1. Conventions for the avalanching region.

drift velocity v_h. In their motion through the semiconductor, electrons and holes generate new electron-hole pairs; this is translated, in statistical terms, by defining the probability density function (p.d.f.) $f_e(\lambda|x)$ and $f_h(\lambda|x)$ that an electron or a hole, respectively, starting at the point x travel along a length λ before generating by ionization a new electron-hole pair.

Our purpose is now to calculate, for an electron or for a hole or for a pair injected at the point x at $t = 0$, the joint p.d.f.'s respectively: $p_e(Q_1, t_1; Q_2, t_2|x)$, $p_h(Q_1, t_1; Q_2, t_2|x)$ and $p(Q_1, t_1; Q_2, t_2|x)$ of observing at

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the output electrode (assumed the one at $x = L$) a charge Q_1 at time t_1 *and* a charge Q_2 at time t_2.

As a first step, let us consider an electron injected at a point x of the avalanche zone at $t = 0$. This electron moves with velocity v_e in the positive x-axis direction and there are two mutually exclusive events for its behaviour. It can drift toward the right hand electrode without producing any ionizing collision in the interval $x - L$; the probability of this event is:

$$1 - \int_x^L f_e(x_1 - x|x)\, dx_1 .$$

Otherwise, the electron will generate the first electron-hole pair at a point x_1 $(x < x_1 < L)$ and the p.d.f. of this event is, by definition, $f_e(x_1 - x|x)$. To obtain the p.d.f. $p_e(Q_1, t_1; Q_2, t_2|x)$ one can therefore sum up the weighted contributions given by each mutually exclusive event:

$$(1) \qquad P_e(Q_1, t_1; Q_2, t_2|x) =$$

$$= P_{NM}(Q_1, t_1; Q_2, t_2|x)\left[1 - \int_x^L f_e(x_1 - x|x)\, dx_1\right] +$$

$$+ \int_x^L P_M(Q_1, t_1; Q_2, t_2|x, x_1)\, f_e(x_1 - x|x)\, dx_1$$

being $P_{NM}(Q_1, t_1; Q_2, t_2|x)$ the joint p.d.f. of the output charges for an electron that, injected at x for $t = 0$, reaches the output electrode without producing any new pair and $P_M(Q_1, t_1; Q_2, t_2|x, x_1)$ the joint p.d.f. of the output charges for an electron that, injected at x for $t = 0$, produces the first pair at the point x_1.

In order to calculate P_{NM} let us denote with $Q_e(t|x)$ the output induced charge due to an electron injected at x for $t = 0$ and moving with constant drift velocity v_e without producing ionization; then the joint p.d.f. $P_{NM}(Q_1, t_1; Q_2, t_2|x)$ is readily seen to be given by:

$$(2) \qquad P_{NM}(Q_1, t_1; Q_2, t_2|x) =$$

$$= \delta[Q_1 - Q_e(t_1|x)]\, \delta[Q_2 - Q_e(t_2|x)]$$

where $\delta(Q)$ is the usual impulse function.

To work out the joint p.d.f. $P_M(Q_1, t_1; Q_2, t_2|x, x_1)$ let us note that it takes the time $\tau = (x_1 - x)/v_e$ for the electron to go from x to x_1, where it produces the first new pair. In the flight from x to x_1 the electron gives an induced output charge whose joint p.d.f. is:

$$\delta[Q_1 - Q_e(t_1|x)\, 1(\tau - t_1)]\, \delta[Q_2 - Q_e(t_2|x)\, 1(\tau - t_2)] .$$

The functions $1(\tau - t_1)$ and $1(\tau - t_2)$ are introduced to limit the validity of this contribution for t_1 and t_2 in the interval $0 - \tau$. At time τ, when the electron is arrived in x_1, we have three carriers. These three carriers may be again considered as the new starting events of the statistical multiplication process.

Since there is statistical independence for the contribution to the output charge, i.e. the total charge observed at the output is the sum of the single contributions, the joint p.d.f. $P_M(Q_1, t_1; Q_2, t_2|x, x_1)$ is the double convolution in Q_1 and Q_2 of the four joint p.d.f.'s associated to each contribution; that is, the first due to electron flight from x to x_1 and those due to the electron and the pair that can be thought of as injected in x_1, at time τ. We can write:

$$(3) \qquad P_M(Q_1, t_1; Q_2, t_2|x, x_1) =$$

$$= \delta[Q_1 - Q_e(t_1|x)\, 1(\tau - t_1)]\, \delta[Q_2 - Q_e(t_2|x)\, 1(\tau - t_2)] *$$

$$* P_e(Q_1, t_1 - \tau; Q_2, t_2 - \tau, x_1) * P_e(Q_1, t_1 - \tau; Q_2, t_2 - \tau, x_1) *$$

$$* P_h(Q_1, t_1 - \tau; Q_2, t_2 - \tau, x_1) .$$

Eq. (1) can therefore be written using eqs. (2) and (3): however, it becomes simpler by introducing the bivariate characteristic functions (b.c.f.) associated to the joint p.d.f.'s, which are the double Fourier transforms of the joint p.d.f.'s, so defined:

$$(4) \qquad \Phi_e(\chi_1, t_1; \chi_2, t_2|x)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_e(Q_1, t_1; Q_2, t_2|x)\, e^{j(\chi_1 Q_1 + \chi_2 Q_2)}\, dQ_1\, dQ_2 .$$

Similar expressions define the b.c.f. associated to $p_h(\ldots)$ and $p(\ldots)$. In this way, transforming both sides of eq. (1) we readily obtain:

$$(5) \qquad \Phi_e(\chi_1, t_1; \chi_2, t_2|x)$$

$$= e^{j\chi_1 Q_e(t|x) + j\chi_2 Q_e(t_2|x)}\left[1 - \int_x^L f_e(x_1 - x|x)\, dx_1\right] +$$

$$+ \int_x^L e^{j\chi_1 Q_e(t_1|x)\, 1(\tau - t_1) + j\chi_2 Q_e(t_2|x)\, 1(\tau - t_2)} \cdot$$

$$\cdot \Phi_e^2(\chi_1, t_1 - \tau; \chi_2, t_2 - \tau|x_1) \cdot$$

$$\cdot \Phi_h(\chi_1, t_1 - \tau; \chi_2, t_2 - \tau|x_1) \cdot f_e(x_1 - x|x)\, dx_1 .$$

The arguments that have led us to eqs. (5) for the b.c.f. relative to one electron injected at time $t = 0$ at x can obviously be applied with slight changes for the case of one hole injected at time $t = 0$ at x. For example, the probability that the hole does not produce ionization while flying with constant saturated velocity v_h from x to 0 is

$$\int_0^x f_h(x - x_1|x)\, dx_1 ;$$

the p.d.f. of producing the first pair in x_1 at time $\theta = (x - x_1)/v_h$ is $f_h(x - x_1|x)$, and so on. Denoting with $Q_h(t|x)$ the charge induced at the output by one hole that moves toward the left hand side electrode without ionizing, one can write rather easily the following equation for the b.c.f. relative to one hole injected at $t = 0$ in x:

$$(6) \qquad \Phi_h(\chi_1, t_1; \chi_2, t_2|x)$$

$$= e^{j\chi_1 Q_h(t_1|x) + j\chi_2 Q_h(t_2|x)}\left[1 - \int_0^x f_h(x - x_1|x)\, dx_1\right] +$$

$$+ \int_0^x e^{j\chi_1 Q_h(t_1|x)\, 1(\theta - t_1) + j\chi_2 Q_h(t_2|x)\, 1(\theta - t_2)} \cdot$$

$$\cdot \Phi_h^2(\chi_1, t_1 - \theta; \chi_2, t_2 - \theta|x_1) \cdot$$

$$\cdot \Phi_e(\chi_1, t_1 - \theta; \chi_2, t_2 - \theta|x_1) \cdot f_h(x - x_1|x)\, dx_1 .$$

Finally, as far as the injection of a pair at x at time $t = 0$ is concerned, it is necessary only to observe that, being independent the contribution given by each carrier the b.c.f. results to be:

$$(7) \qquad \Phi(\chi_1, t_1; \chi_2, t_2|x) = \Phi_e(\chi_1, t_1; \chi_2, t_2|x)\, \Phi_h(\chi_1, t_1; \chi_2, t_2|x)\,.$$

The system of integral equations (5) and (6), enables one to obtain, at least in principle, the b.c.f. $\Phi_e(\chi_1, t_1; \chi_2, t_2|x)$ and $\Phi_h(\chi_1, t_1; \chi_2, t_2|x)$ which characterize completely, at second order statistics, the behaviour of the avalanching photodiode.

Moreover, the restriction of a charge carrier injected for $t = 0$ at x can be removed easily: if $p(x, t)$ is the p.d.f. of a generical injection at point x and at time t, for example of pairs, the b.c.f. $\Phi_i(\chi_1, t_1; \chi_2, t_2)$ of the charge observed at the output results:

$$(8) \qquad \Phi_i(\chi_1, t_1; \chi_2, t_2) = \int_0^L \int_0^\infty \Phi(\chi_1, t_1 - \sigma; \chi_2, t_2 - \sigma|x)\, p(x, \sigma)\, dx\, d\sigma\,.$$

And finally if R charge carriers are injected each one with the joint p.d.f. $p(x, t)$ the b.c.f. $\Phi_{iR}(\chi_1, t_1; \chi_2, t_2)$ of the charge observed at the output becomes:

$$(8') \qquad \Phi_{iR}(\chi_1, t_1; \chi_2, t_2) = [\Phi_i(\chi_1, t_1; \chi_2, t_2)]^R\,.$$

THE INTEGRAL EQUATIONS FOR MEAN AND AUTOCORRELATION FUNCTIONS OF THE OUTPUT PULSE.

The advantage of introducing the b.c.f.'s is now readily appreciated. In fact, if we want to calculate the mean output observed charge $q_e(t|x)$ due to one electron injected at x for $t = 0$ and its autocorrelation function $\Gamma_e(t_1, t_2|x)$, we need only to apply, as it is well known and easily checked by the definition of b.c.f. (4) the following equations:

$$(9) \qquad q_e(t_1|x) = \left.\frac{\partial \Phi_e(\chi_1, t_1; \chi_2, t_2|x)}{\partial \chi_1}\right|_{\chi_1 = \chi_2 = 0},$$

$$(10) \qquad \Gamma_e(t_1, t_2|x) = \left.\frac{\partial^2 \Phi_e(\chi_1, t_1; \chi_2, t_2|x)}{\partial \chi_1\, \partial \chi_2}\right|_{\chi_1 = \chi_2 = 0}.$$

Similar equations, which can be obtained only changing the subscripts, hold for the means $q_h(t_1|x)$, $q(t_1|x)$ and the autocorrelation functions $\Gamma_h(t_1, t_2|x)$, $\Gamma(t_1, t_2|x)$ of the output observed charge for one hole or one pair respectively injected at x, for $t = 0$.

Accordingly, in view of eq. (7) one can obtain at once, the following relations:

$$(11) \qquad q(t|x) = q_e(t|x) + q_h(t|x)$$

$$(12) \qquad \Gamma(t_1, t_2|x) = \Gamma_e(t_1, t_2|x) + \Gamma_h(t_1, t_2|x) + q_e(t_1|x)\, q_h(t_2|x) + q_e(t_2|x)\, q_h(t_1|x)$$

and, as far as eq. (8) is concerned, we have also:

$$(13) \qquad q_i(t) = \int_0^L \int_0^\infty q(t - \sigma|x)\, p(x, \sigma)\, dx\, d\sigma$$

$$(14) \qquad \Gamma_i(t_1, t_2) = \int_0^L \int_0^\infty \Gamma(t_1 - \sigma, t_2 - \sigma|x)\, p(x, \sigma)\, dx\, d\sigma\,.$$

By introducing eqs. (5) and (6) in eqs. (9) and (10) we obtain the following simultaneous equations for the means and the autocorrelation functions of the charge observed at the output:

$$(15) \qquad q_e(t|x) = Q_e(t|x) + \int_x^L [2\, q_e(t - \tau|x_1) + q_h(t - \tau|x_1) - Q_e(t - \tau|x_1)] \cdot f_e(x_1 - x|x)\, dx_1$$

$$(15') \qquad q_h(t|x) = Q_h(t|x) + \int_0^x [2\, q_h(t - \theta|x_1) + q_e(t - \theta|x_1) - Q_h(t - \theta|x_1)] \cdot f_h(x - x_1|x)\, dx_1$$

$$(16) \qquad \Gamma_e(t_1, t_2|x) = q_e(t_1|x)\, Q_e(t_2|x) + q_e(t_2|x)\, Q_e(t_1|x) - Q_e(t_1|x)\, Q_e(t_2|x) + \int_x^L \{2\, \Gamma_e(t_1 - \tau, t_2 - \tau|x_1) + \Gamma_h(t_1 - \tau, t_2 - \tau|x_1) + 2\, q_e(t_1 - \tau|x_1) \cdot [q_e(t_2 - \tau|x_1) + q_h(t_2 - \tau|x_1) - Q_e(t_2 - \tau|x_1)] + q_h(t_1 - \tau|x_1)\, [2\, q_e(t_2 - \tau|x_1) - Q_e(t_2 - \tau|x_1)] + Q_e(t_1 - \tau|x_1)\, [Q_e(t_2 - \tau|x_1) - 2\, q_e(t_2 - \tau|x_1) - q_h(t_2 - \tau|x_1)]\}\, f_e(x_1 - x|x)\, dx_1$$

$$(16') \qquad \Gamma_h(t_1, t_2|x) = q_h(t_1|x)\, Q_h(t_2|x) + q_h(t_2|x)\, Q_h(t_1|x) - Q_h(t_1|x)\, Q_h(t_2|x) + \int_0^x \{2\, \Gamma_h(t_1 - \theta, t_2 - \theta|x_1) + \Gamma_e(t_1 - \theta, t_2 - \theta|x_1) + 2\, q_h(t_1 - \theta|x_1) \cdot [q_h(t_2 - \theta|x_1) + q_e(t_2 - \theta|x_1) - Q_h(t_2 - \theta|x_1)] + q_e(t_1 - \theta|x_1)\, [2\, q_h(t_2 - \theta|x_1) - Q_h(t_2 - \theta|x_1)] + Q_h(t_1 - \theta|x_1)\, [Q_h(t_2 - \theta|x_1) - 2\, q_h(t_2 - \theta|x_1) - q_e(t_2 - \theta|x_1)]\}\, f_h(x - x_1|x)\, dx_1\,.$$

We assume now the following expressions for the p.d.f.'s $f_e(x_1 - x|x)$ and $f_h(x - x_1|x)$ of the length $x_1 - x$ between ionizations:

$$(17) \qquad f_e(x_1 - x|x) = \alpha\, e^{-\alpha(x_1 - x)}$$

$$(18) \qquad f_h(x - x_1|x) = \beta\, e^{-\beta(x - x_1)}$$

where α and β, the ionization rates for unity length, are eventually functions of x. Eqs. (17) and (18) imply that the number of ionization in a fixed length is Poisson-distributed.

The solution of the integral systems (15) (15') and (16) (16') is greatly simplified by using Laplace transforms for the time variables (the bivariate one in the second case); therefore we get the new integral systems in the frequency variables p, p_1 and p_2 corresponding to t, t_1

and t_2:

$$(19)\qquad q_e(p|x) = Q_e(p|x) - \alpha\, e^{\left(\alpha + \frac{p}{v_e}\right)x} \int_x^L [2\, q_e(p|x_1) + q_h(p|x_1) - Q_e(p|x_1)]\, e^{-\left(\alpha + \frac{p}{v_e}\right)x_1}\, dx_1$$

$$(19')\qquad q_h(p|x) = Q_h(p|x) + \beta\, e^{-\left(\beta + \frac{p}{v_h}\right)x} \int_0^x [2\, q_h(p|x_1) + q_e(p|x_1) - Q_h(p|x_1)]\, e^{\left(\beta + \frac{p}{v_h}\right)x_1}\, dx_1$$

$$(20)\qquad \Gamma_e(p_1, p_2|x) = q_e(p_1|x)\, Q_e(p_2|x) + q_e(p_2|x)\, Q_e(p_1|x) - Q_e(p_1|x)\, Q_e(p_2|x) + \alpha\, e^{\left(\alpha + \frac{p_1 + p_2}{v_e}\right)x} \int_x^L \{2\, \Gamma_e(p_1, p_2|x_1) + \Gamma_h(p_1, p_2|x_1) + 2\, q_e(p_1|x_1)\, [q_e(p_2|x_1) + q_h(p_2|x_1) - Q_e(p_2|x_1)] + q_h(p_1|x_1)\, [2\, q_e(p_2|x_1) - Q_e(p_2|x_1)] + Q_e(p_1|x_1)\, [Q_e(p_2|x_1) - 2\, q_e(p_2|x_1) - q_h(p_2|x_1)]\} \cdot e^{-\left(\alpha + \frac{p_1 + p_2}{v_e}\right)x_1}\, dx_1 .$$

$$(20')\qquad \Gamma_h(p_1, p_2|x) = q_h(p_1|x)\, Q_h(p_2|x) + q_h(p_2|x)\, Q_h(p_1|x) - Q_h(p_1|x)\, Q_h(p_2|x) - \beta\, e^{-\left(\beta + \frac{p_1 + p_2}{v_h}\right)x} \int_0^x \{2\, \Gamma_h(p_1, p_2|x_1) + \Gamma_e(p_1, p_2|x_1) + 2\, q_h(p_1|x_1)\, [q_h(p_2|x_1) + q_e(p_2|x_1) - Q_h(p_2|x_1)] + q_e(p_1|x_1)\, [2\, q_h(p_2|x_1) - Q_h(p_2|x_1)] + Q_h(p_1|x)\, [Q_h(p_2|x_1) - 2\, q_h(p_2|x_1) - q_e(p_2|x_1)]\} \cdot e^{\left(\beta + \frac{p_1 + p_2}{v_h}\right)x_1}\, dx_1 .$$

MEAN OUTPUT VALUE.

The system of integral equation given by (19) and (19′) can be solved leading to a second order linear differential equation with constant coefficients in the x variable; we obtain:

$$(21)\qquad \frac{\partial^2 q_e(p|x)}{\partial x^2} + \left(\alpha - \frac{p}{v_e} - \beta + \frac{p}{v_h}\right) \frac{\partial q_e(p|x)}{\partial x} + \left(\frac{\alpha\, p}{v_h} + \frac{\beta\, p}{v_e} - \frac{p^2}{v_e\, v_h}\right) q_e(p|x) = - \frac{x}{L}\left(\frac{\alpha}{v_h} - \frac{\beta}{v_e} - \frac{p}{v_e\, v_h}\right) - \frac{1}{v_e\, L}$$

$$(21')\qquad q_h(p|x) = -\frac{1}{\alpha}\left\{\frac{\partial q_e(p|x)}{\partial x} - \left(\alpha - \frac{p}{v_e}\right) q_e(p|x) + \frac{x}{v_e\, L}\right\} .$$

We have also utilized, for the induced charges $Q_e(p|x)$ and $Q_h(p|x)$ the following expressions:

$$(22)\qquad Q_e(p|x) = \frac{x}{p\, L} + \frac{v_e}{p^2\, L}\left[1 - e^{-p\frac{L - x}{v_e}}\right]$$

$$(22')\qquad Q_h(p|x) = \frac{x}{p\, L} + \frac{v_h}{p^2\, L}\left[1 - e^{-p\frac{x}{v_h}}\right].$$

Eqs. (21) and (21′) are to be solved with the simple boundary conditions $q_e(p|L) = 1/p$, $q_h(p|0) = 0$ already contained in eqs. (19) and (19′). Therefore we get as a final result:

$$(23)\qquad q_e(p|x) = \frac{A + x}{p\, L} + B\, e^{\xi_1 x} - C\, e^{\xi_2 x}$$

where ξ_1 and ξ_2 are the roots of the characteristic equation associated to (21):

$$\xi_{1,2} = \frac{\beta - \alpha}{2} + \frac{p}{2}\left(\frac{1}{v_e} - \frac{1}{v_h}\right) \pm \left[\frac{1}{4}\left(\alpha - \beta - \frac{p}{v_e} - \frac{p}{v_h}\right)^2 - \alpha\, \beta\right]^{1/2}$$

and A, B, C are constants given by:

$$A = \frac{\xi_1 + \xi_2 - p/v_e}{\xi_1\, \xi_2}$$

$$B = -\frac{A\, (\xi_2 - \alpha - p/v_e)\, e^{-\xi_1 L} + [(\alpha - p/v_e)\, A + 1]\, e^{(\xi_2 - \xi_1) L}}{p\, L\, [\xi_2 + \alpha - p/v_e - (\xi_1 + \alpha - p/v_e)\, e^{(\xi_2 - \xi_1) L}]}$$

$$C = B\, e^{(\xi_1 - \xi_2) L} - \frac{A}{p\, L}\, e^{-\xi_2 L} .$$

One can then calculate $q_h(p|x)$ by simply applying eq. (21′) and finally obtains, for $q(p|x)$, using eq. (11) also:

$$(24)\qquad q(p|0) = \frac{1}{\alpha}\left\{\left(\frac{p}{v_e} - \xi_1\right) B\, e^{\xi_1 x} - \left(\frac{p}{v_e} - \xi_2\right) C\, e^{\xi_2 x} - \frac{1}{L}\left(\frac{A}{v_e} - \frac{1}{p}\right)\right\} .$$

Eq. (23) and eq. (24) solve the problem of the mean response of the avalanche diode when excited at a point

x by an electron or a pair. The statistical method used enables us to obtain in this way not only the output effect due to the simultaneous injection of a pair, but also the one due to the injection of a single carrier, whether electron or hole. The output current pulse for the various types of injection considered has a mean value that is readily obtainable, from eqs. (23) and (24) by multiplication by p:

$$i_e(p|x) = p\, q_e(p|x) \tag{25}$$

$$i\,(p|x) = p\, q\,(p|x)\,. \tag{25'}$$

From the general expressions (25) and (25′) we can obtain some interesting quantities more directly characterizing the avalanching diodes.

a) Multiplication factor.

From the limit theorem of the Laplace transform we can directly obtain, taking the limit for $p \to 0$ in eq. (25) and (25′), the multiplication factors i.e., the mean total number of electrons or of pairs for one electron or pair injected at point x:

$$M_e(x) = i_e(0|x) = \frac{\beta - \alpha\, e^{(\beta-\alpha)x}}{\beta - \alpha\, e^{(\beta-\alpha)L}} \tag{26}$$

$$M\,(x) = i\,(0|x) = \frac{(\beta-\alpha)\, e^{(\beta-\alpha)L}}{\beta - \alpha\, e^{(\beta-\alpha)L}} \tag{27}$$

and also, as a consequence of eq. (11) one has: $M_h(x) = M(x) - M_e(x)$ for the hole multiplication factor. In the case in which the pairs are injected with p.d.f. $p(x)$ eq. (13) gives readily:

$$M_i = \int_0^L M(x)\, p(x)\, dx\,. \tag{28}$$

Expressions (26), (27) and (28) can be also much easier obtained using directly the differential equations (21) and (21′) and taking into account eqs. (25) and (25′). It is interesting to observe, that being well known [2, 3] the pair multiplication factor given by (26), the result due to the injection of one single carrier given by (27) seems to us new.

b) Frequency response of the avalanching diode.

Letting $p = j\omega$ in (25′) we obtain the frequency response for pair injection at point x. Though eq. (25′) is rather involved, in two limit cases it is possible to describe analitically with good accuracy the frequency spectrum of the mean current pulse at the output $i\,(j\omega|x)$.

For $\alpha = \beta$ by developing in series ξ_1 and ξ_2 at the first order in

$$\frac{\omega L}{v^*} = \omega L\left(\frac{1}{v_e} + \frac{1}{v_h}\right)$$

(i.e. in the hypothesis $\omega L/v^* \ll 1$) one can get that, for multiplication factors $M(x) \gg 1$:

$$i\,(j\omega|x) = \frac{M(x)}{1 + j\,\dfrac{\omega L\, M(0)}{6\, v^*}}\,. \tag{29}$$

The resulting cutoff frequency given by

$$f_T = \frac{3\, v^*}{\pi L\, M(0)}\,,$$

is accurate for

$$\frac{6\, v^*}{L\, M(0)} \ll \frac{v^*}{L}\,,$$

that is for $M(0) \gg 6$.

In this case, if the injection of pairs has the joint p.d.f. $p(t|x)$ whose Laplace transform is $H(p|x)$, then, by eq. (13):

$$i_i(j\omega) = \frac{\displaystyle\int_0^L M(x)\, H(j\omega|x)\, dx}{1 + j\,\dfrac{\omega L\, M(0)}{6\, v^*}}\,. \tag{30}$$

In the opposite case that is for $\alpha \gg \beta$, one can again find, by developing at the first order ξ_1, and ξ_2 in $\omega/\alpha v^*$, that for $M(x) \gg 1$

$$i\,(j\omega|x) = \frac{M(x)}{1 + j\omega\, M(0)\, \dfrac{\beta\,(\alpha L - 2)}{\alpha^2 v^*}}\,. \tag{31}$$

The corresponding cutoff frequency is

$$f_T = \frac{\alpha^2 v^*}{2\pi\, M(0)\, \beta\,(\alpha L - 2)}$$

and is accurate for $\alpha/\beta \ll M(0)\,(\alpha L - 2)$ that is, for $M(0) \gg 1$, so that $\alpha L \simeq \log \alpha/\beta$ and, therefore, when $M(0) \gg (\alpha/\beta)/(\log \alpha/\beta - 2)$.

Moreover, one can verify from eq. (13) that for an injection of pairs with joint p.d.f. $p(t|x)$ the corresponding $i_i(j\omega)$ is:

$$i_i(j\omega) = \frac{\displaystyle\int_0^L M(x)\, H(j\omega|x)\, dx}{1 + j\omega\, M(0)\, \dfrac{\beta\,(\alpha L - 2)}{\alpha^2 v^*}}\,. \tag{32}$$

Expressions similar to (29), (31) except for minor differences have been obtained, through a different analysis, by Chang [4].

The analitical expressions (29) and (31) for the transfer function, do no cover many interesting cases being their validity generally limited to large gains $M(0)$. So, for the general case we have obtained through a digital computer the spectrum $|\,i\,(j\omega\,|\,0)\,|$ of the mean current observed at the output for one pair injected at $x = 0$, obviously it is $|\,i\,(j\omega|0)| = |\,i_e(j\omega|0)\,|$ for various values of the multiplication factor $M(0)$ and of the ratio β/α between the ionization rates, as a function of the standardized frequency $\omega L/v^*$; it has been assumed $v_e = v_h$. Figs. 2, 3, 4 and 5 show

$$|\,i\,(j\omega|0)\,| \ \text{vs}\ \frac{\omega L}{v^*}$$

for $\beta/\alpha = 1;\ 0{,}1;\ 0{,}01;\ 0$ respectively.

Fig. 6 shows the behaviour of the standardized cutoff frequency $\omega_T L/v^*$ vs the ratio α/β for various values of the gain M (o).

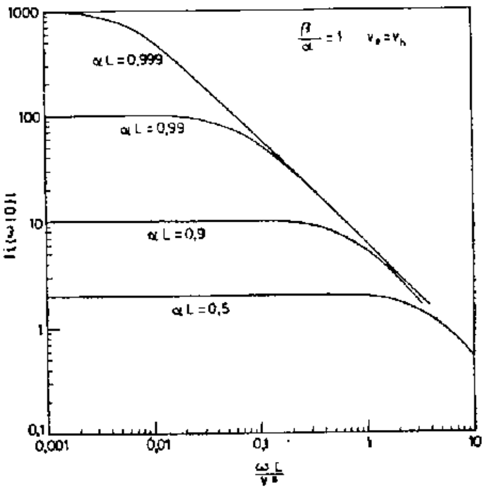

Fig. 2. — The frequency response of the mean output current due to pair injection at $x = 0$ for various multiplication factors M (o); $\beta/\alpha = 1$.

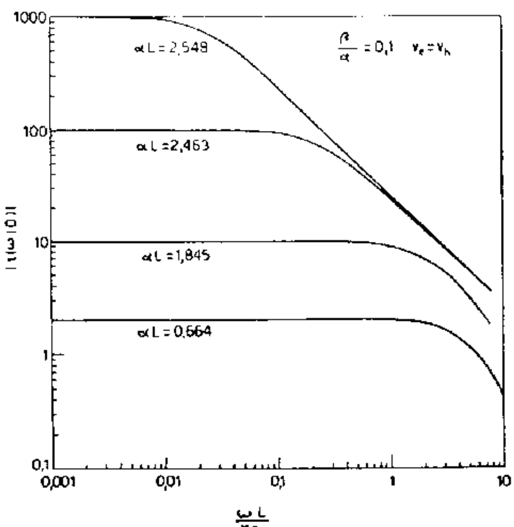

Fig. 3. — The same as fig. 2, with $\beta/\alpha = 0,1$.

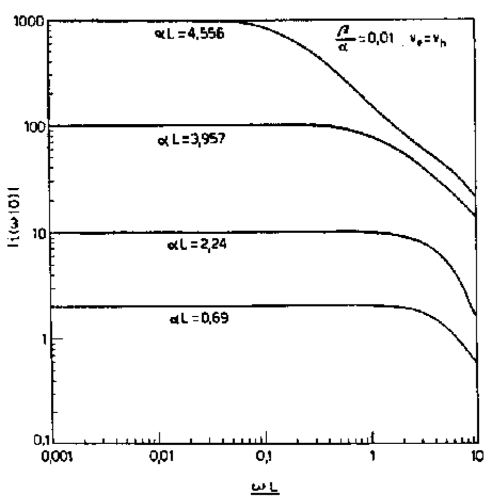

Fig. 4. — The same as fig. 2, with $\beta/\alpha = 0,01$.

c) The single carrier response.

The frequency response fully characterizes the behaviour of the avalanching photodiode but, for the electronic handlings, it is also interesting an insight on the time domain features of the mean output current

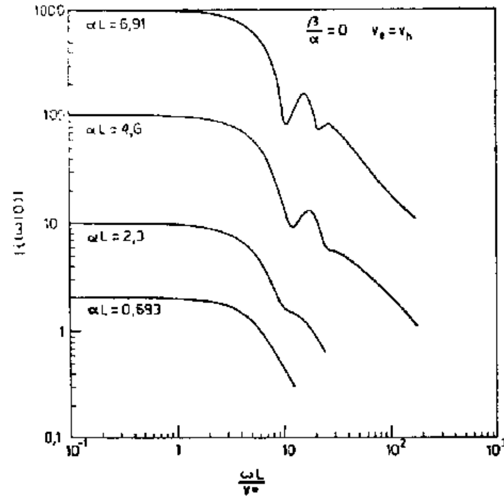

Fig. 5. — The same as fig. 2, with $\beta/\alpha = 0$.

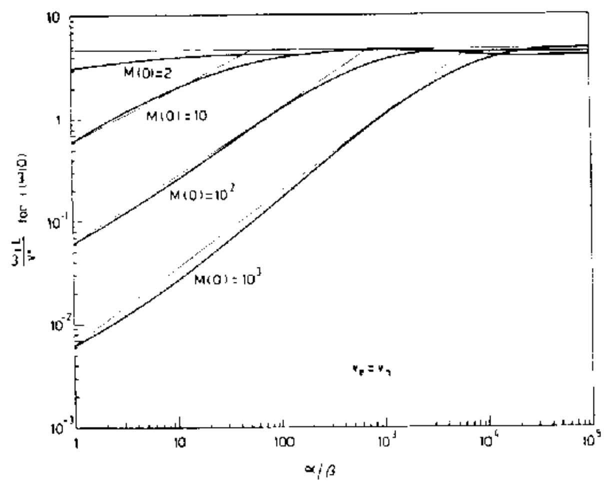

Fig. 6. — The standardized cutoff frequency of the mean output current due to pair injection at $x = 0$ for various M (o).

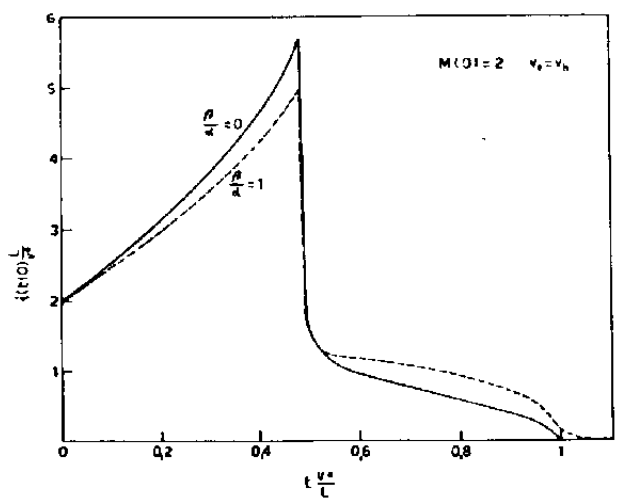

Fig. 7. — The mean output current pulse $i(t|0)$ for multiplication factor M (o) = 2.

pulse $i\,(t|x)$. Letting $p = j\,\omega$ in (25′), it is straightforward, by taking the inverse Fourier transform, to obtain $i\,(t|x)$. We have limited ourselves to the interesting case

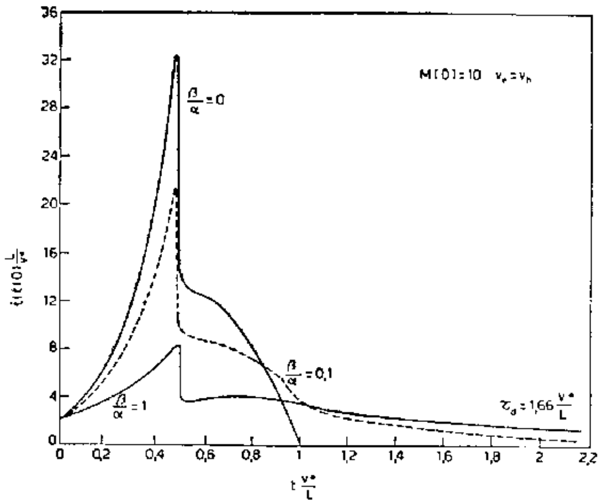

Fig. 8. — The same as fig. 7, with M (o) = 10.

ionization; when $\beta/\alpha \neq$ o and especially for $\beta/\alpha > 1/M$ (o), the shape of the output current pulse changes and besides the fast rise there appears a slow tail due to the positive feedback mechanism of multiplication. It is interesting to note that the sharp decrease of the current pulse at time L/v_e suggests promising facilities in time measurements with avalanching photodiodes.

Autocorrelation functions.

As for the mean values, the integral equations (20) and (20′) can be solved through a second order differential equation with constant coefficients in the x variable. To obtain a more compact form for the differential equation we can introduce the following notation

(33)

$$\Gamma_{ea}\,(p_1, p_2|x) = \Gamma_e\,(p_1, p_2|x) - q_e\,(p_1|x)\,Q_e\,(p_2|x) - $$
$$- q_e\,(p_2|x)\,Q_e\,(p_1|x) - Q_e\,(p_1|x)\,Q_e\,(p_2|x)\,,$$
$$\Gamma_{ha}\,(p_1, p_2|x) = \Gamma_h\,(p_1, p_2|x) - q_h\,(p_1|x)\,Q_h\,(p_2|x) -$$
$$- q_h\,(p_2|x)\,Q_h\,(p_1|x) - Q_h\,(p_1|x)\,Q_h\,(p_2|x)\,.$$

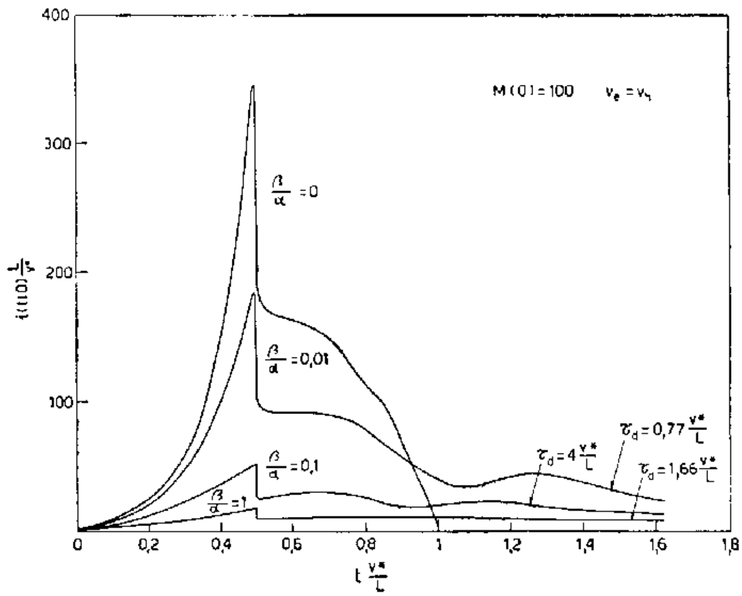

Fig. 9. — The same as fig. 7, with M (o) = 100.

$x =$ o and calculated, with an electronic computer, $i\,(t|$o) for different values of M (o) and β/α. A fast Fourier transform program [8] has been utilized, in which care has been taken to smooth Gibbs oscillations around discontinuities.

Figs. 7, 8 and 9 show $i\,(t|$o) L/v^* against the standardized time $t\,v^*/L$ for M (o) = 2, 10, 100 respectively, having assumed equal hole and electron velocities $v_e = v_h$.

Every diagram starts, at $t =$ o from the value $2\,v^*/L = v_e/L$ which is the electron induced charge contribution lasting for the time $L/2\,v^* = L/v_e$. In the time interval o $\div\ L/v_e$, $i\,(t|$o) increases exponentially because of the avalanche build up; at time $t = L/v_e$ all the electrons following electrons ionizations are collected at the output as they move with the same velocity v_e and therefore there is a step-like decrease of $i\,(t|$o). In the case $\beta/\alpha =$ o the current ceases for $t = L/v^*$ as there is no holes

By utilizing (33), from (20) and (21′) we get:

(34)
$$-\frac{\partial^2 \Gamma_{ea}\,(p_1, p_2|x)}{\partial x^2} +$$
$$+ \left[\alpha - \frac{p_1 + p_2}{v_e} - \beta + \frac{p_1 + p_2}{v_h}\right] \frac{\partial \Gamma_{ea}\,(p_1, p_2|x)}{\partial x} +$$
$$+ \left[\frac{\alpha\,(p_1 + p_2)}{v_h} - \frac{\beta\,(p_1 + p_2)}{v_e} - \frac{(p_1 + p_2)^2}{v_e\,v_h}\right] \Gamma_{ea}\,(p_1, p_2|x) -$$
$$= - \alpha\,\frac{\partial H_1\,(p_1, p_2|x)}{\partial x} - \alpha\,\frac{p_1 + p_2}{v_h}\,H_1\,(p_1, p_2|x) +$$
$$+ 2\,\alpha\,\beta\,[q_e\,(p_1|x)\,q_e\,(p_2|x) - q_h\,(p_1|x)\,q_h\,(p_2|x)] +$$
$$+ \alpha\,\beta\,H_2\,(p_1, p_2|x)$$

where:

(35)

$$H_1(p_1, p_2|x) = 2\,q_e(p_1|x)\,q_e(p_2|x) -$$
$$- Q_e(p_1|x)\,Q_e(p_2|x) - Q_h(p_1|x)\,Q_h(p_2|x) +$$
$$+ q_h(p_1|x)\,[2\,q_e(p_2|x) + Q_h(p_2|x) - Q_e(p_2|x)] +$$
$$+ q_h(p_2|x)\,[2\,q_e(p_1|x) + Q_h(p_1|x) - Q_e(p_1|x)]\,,$$
$$H_2(p_1, p_2|x) = q_h(p_1|x)\,[Q_h(p_2|x) - Q_e(p_2|x)] +$$
$$+ q_h(p_2|x)\,[Q_h(p_1|x) - Q_e(p_1|x)] -$$
$$- q_e(p_1|x)\,[Q_e(p_2|x) - Q_h(p_2|x)] -$$
$$- q_e(p_2|x)\,[Q_e(p_1|x) - Q_h(p_1|x)]\,.$$

Moreover, $\Gamma_{ha}(p_1, p_2|x)$ is given by:

(36) $$\Gamma_{ha}(p_1, p_2|x) =$$
$$= -\frac{1}{\alpha}\left[\frac{\partial \Gamma_{ea}(p_1, p_2|x)}{\partial x} + \left(\alpha - \frac{p_1 + p_2}{v_e}\right)\Gamma_{ea}(p_1, p_2|x)\right] -$$
$$- H_1(p_1, p_2|x)\,.$$

These differential equations must be solved with the boundary conditions, already contained in eqs. (20) and (20'):

(37) $$\Gamma_{ea}(p_1, p_2|L) = 0\,, \qquad \Gamma_{ha}(p_1, p_2|0) = 0\,.$$

The solution of eq. (34), taking also into account (35), (36) and (33) fully characterizes the second order statistical properties of the output charge of the avalanche diode. However, by knowing the autocorrelation function transforms for the output charge $\Gamma_e(p_1, p_2|x)$, $\Gamma(p_1, p_2|x)$ we can obtain also the autocorrelation function for the output current $A_e(p_1, p_2|x)$ and $A(p_1, p_2|x)$ due respectively to an electron or a pair injected at x for $t = 0$; we have:

(38)
$$A_e(p_1, p_2|x) = p_1\,p_2\,\Gamma_e(p_1, p_2|x)$$
$$A(p_1, p_2|x) = p_1\,p_2\,\Gamma(p_1, p_2|x)\,.$$

Moreover, if $g(x, t)$ is the joint p.d.f. of the excitation at point x and time t, from eq. (8) we can obtain for the autocorrelation function of the output current the following expression:

$$A_i(p_1, p_2) = \int_0^L A(p_1, p_2|x)\,G(x, p_1 + p_2)\,dx$$

where $G(x, p)$ is the Laplace transform of $g(x, t)$.

An analitical expression in the general case, for the solution of (34) is very cumbersome. We can obtain, as particular cases, some interesting results as the photodiode gain variance $\varepsilon^2(x)$ and the current power spectrum.

a) Photodiode gain variance $\varepsilon^2(x)$.

It is easily shown that the limit for $p_1 = p_2 = 0$ of the current autocorrelation function $A(p_1, p_2|x)$ is the initial second moment of the random variable total output charge. That is, we have:

$$A(0, 0|x) = \varepsilon^2(x) + M^2(x)\,.$$

So, by integrating eq. (34) with $p_1 = p_2 = 0$ we obtain for the variance $\varepsilon^2(x)$ of the total number of pairs following one pair injection at point x:

(39) $$\varepsilon^2(x) =$$
$$= M^2(x) + \frac{\beta}{\alpha - \beta}\,M^2(0)\,M(x) - \frac{\alpha}{\alpha - \beta}\,M^2(L)\,M(x)$$

The integration yelds also the variance $\varepsilon_e^2(x)$ of the total number of electrons following one electron injection at point x:

(40) $$\varepsilon_e^2(x) =$$
$$= \frac{\alpha}{\alpha - \beta}\,M^2(x) + \frac{\alpha\,\beta}{(\alpha - \beta)^2}\,M^2(0)\,M(x) -$$
$$- \frac{\alpha^2}{(\alpha - \beta)^2}\,M^2(L)\,M(x) + \frac{\alpha\,\beta}{(\alpha - \beta)^2}\,[M^2(L)\,M(0) -$$
$$- M^2(0)\,M(L)]\,.$$

Expression (39) has been already obtained, in a different way, by Ogawa [2] and McIntyre [3].

b) Current Power Spectrum.

It is well known that the power spectral density of a stationary random function is the Fourier transform of the autocovariance function. From eq. (38) one can get the autocovariance function of the output current pulse from the avalanche diode for every type of excitation. Let us consider the problem of finding the power spectral density of the output current noise when the avalanche diode is excited by a sequence, Poisson-distributed in time, of pairs at the point x, being F the mean frequency of excitation. This case, for $x = 0$, corresponds to the case of output noise power spectrum due to minority carriers injected at one side of the multiplying structure. It is shown in Appendix that the noise power spectrum is given, in this case, in terms of the autocorrelation function by:

(41) $$S(\omega|x) = 2\,F\,A(j\,\omega, -j\,\omega|x) \qquad (0 < \omega < \infty)$$

Correspondingly, the differential equations (34) are greatly simplified and one gets by two integrations, for $x = 0$:

(42) $$S(\omega|0) = 2\,F\left\{M(0) + \right.$$
$$+ \int_0^L \left[f_1(\omega, x) + e^{(\beta - \alpha)x}\int_0^x e^{(\alpha - \beta)x'}\,f_2(\omega, x')\,dx'\right]dx +$$
$$\left. + 2\,\mathrm{Re}\,[I_h(j\,\omega|0)\,I_e(-j\,\omega|0)] - |\,I_e(j\,\omega|0)\,|^2\right\}$$

where:

$$I_e(j\,\omega|x) = j\,\omega\,Q_e(j\,\omega|x)\,, \qquad I_h(j\,\omega|x) = j\,\omega\,Q_h(j\,\omega|x)$$

$$f_1(\omega, x) =$$
$$= \alpha\,\{2\,|\,I_e(j\,\omega|x)\,|^2 + 4\,\mathrm{Re}\,[I_e(j\,\omega|x)\,I_h(-j\,\omega|x)] +$$
$$+ 2\,\mathrm{Re}\,[I_h(j\,\omega|x)\,(I_e(-j\,\omega|x) - I_h(-j\,\omega|x))]\} +$$
$$+ |\,I_e(j\,\omega|x)\,|^2 - |\,I_h(j\,\omega|x)\,|^2\,;$$

$$f_2(\omega, x) =$$
$$= 2\,\alpha\,\beta\,\{|\,i_h(j\,\omega|x)\,|^2 + |\,i_e(j\,\omega|x)\,|^2 +$$
$$+ \mathrm{Re}\,[i_e(j\,\omega|x)\,(I_e(-j\,\omega|x) + I_h(j\,\omega|x))]\} +$$
$$+ \alpha\,(\beta - \alpha)\,f_1(\omega, x)\,.$$

Also for the power spectrum the analytical expression for the two limit cases $\alpha = \beta$ and $\alpha \gg \beta$ are very simple; one can obtain, in the same hypotesis assumed for the mean value approximations:

$$(43)\qquad s(\omega|x) = 2\,F\,\frac{\varepsilon^2(x) + M^2(x)}{1 + \left[\dfrac{\omega\,L\,M(0)}{6\,v^*}\right]^2}$$
$$\left[\alpha = \beta,\ \frac{\omega\,L}{v^*} \ll 1,\ M(x) \gg 1\right]$$

$$(44)\qquad s(\omega|x) = 2\,F\,\frac{\varepsilon^2(x) + M^2(x)}{1 + \left[\dfrac{\omega\,M(0)\,\beta\,(\alpha\,L - 2)}{\alpha^2\,v^*}\right]^2}$$
$$\left[\alpha \gg \beta,\ \frac{\omega}{\alpha\,v^*} \ll 1,\ M(x) \gg 1\right]$$

In the general case we have obtained, through a digital computer, $s(\omega|0)$ as a function of the standardized frequency $\omega\,L/v^*$ for various values of the ratio β/α of the ionization rates and of the multiplication factor $M(0)$. Figs. 10, 11, 12 and 13 show $s(\omega|0)$ vs $\omega\,L/v^*$

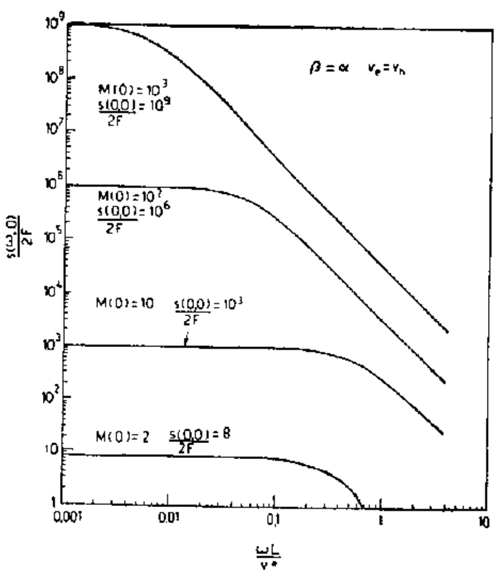

Fig. 10. — The noise power spectrum of output current pulse due to pair injection at $x = 0$ for various $M(0)$ and $\beta/\alpha = 1$.

for $\beta/\alpha = 1$; 0,1; 0,01; 0 and $M(0) = 2$; 10; 10^2; 10^3 having assumed equal hole and electron velocities $v_e = v_h$. The standardized cutoff frequency $\omega_T\,L/v^*$ (6 dB point) of the noise power spectrum is shown, as a function of α/β, with $M(0)$ as a parameter, in fig. 14.

From figs. 6 and 14 it is possible to verify that the cutoff frequencies for the spectrum of the mean output current pulse and for the noise power spectrum are practically equal for multiplication factors $M(0) \geq 10$; also

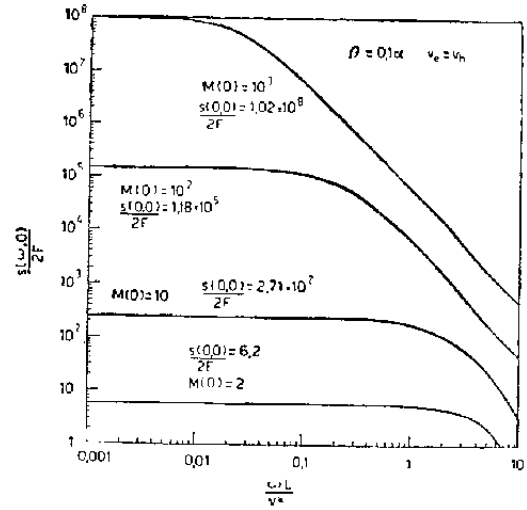

Fig. 11. — The same as fig. 10, with $\beta/\alpha = 0{,}1$.

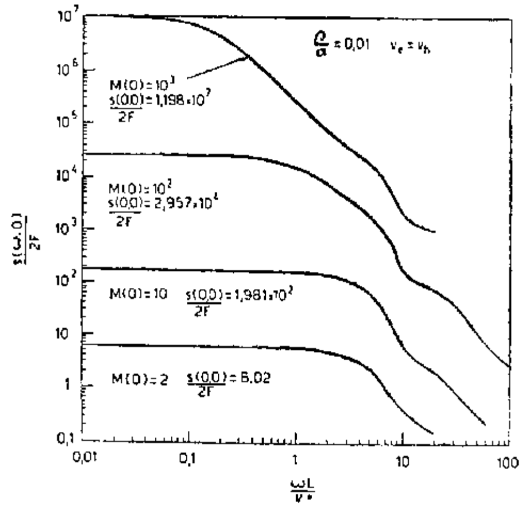

Fig. 12. — The same as fig. 10, for $\beta/\alpha = 0{,}01$.

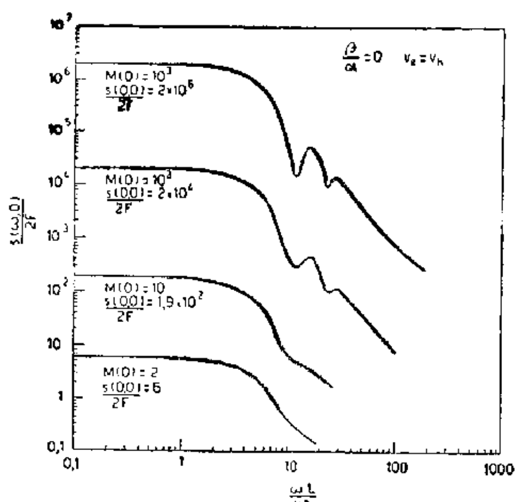

Fig. 13. — The same as fig. 10, for $\beta/\alpha = 0$.

for $M(0) = 2$ the cutoff frequency of the noise is only about 20% higher than that of the signal.

Moreover it is possible to describe fairly well, analitically, the cutoff standardized frequency, with the following formula which holds within 20%:

$$\frac{\omega_T L}{v^*} = a \qquad \text{where } a = \frac{6}{M(0)}\left(\frac{\alpha}{\beta}\right)^{\frac{\lg M(0)}{1+\lg M(0)}} \qquad \text{for } a < 4{,}5$$

(45)

$$\frac{\omega_T L}{v^*} = 4{,}5 \qquad \text{for } a \geqslant 4{,}5$$

This approximation is shown, in light lines, in figs. 6 and 14.

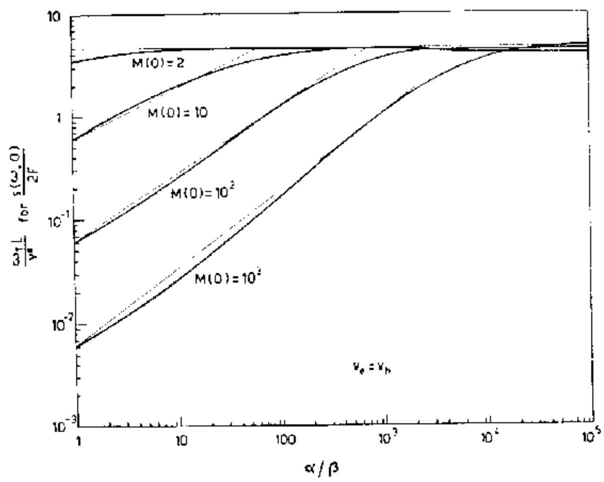

Fig. 14. — The standardized cutoff frequency of the noise power spectrum due to pair injection at $x = 0$ for various $M(0)$.

CONCLUSIONS.

We have already observed that the cutoff frequency of the mean response and of the noise are practically coincident. Moreover, by comparing the frequency spectrum of the mean current $|i(j\omega|0)|$ shown in figs. 2, 3, 4 and 5 with the corresponding one for the noise power spectrum $s(\omega|0)$ of figs. 10, 11, 12 and 13, it is possible to see that the following relation holds with good approximation:

$$s(\omega|0) = 2F\left[1 + \frac{\varepsilon^2(0)}{M^2(0)}\right]|i(j\omega|0)|^2 .$$

The approximation is good also for frequencies much higher than the cutoff frequency, mainly when there is a large positive feedback, that is for $M(0) > \alpha/\beta$.

This property can be expressed, in the time domain, by saying that the current response of the avalanche diode to the single carrier is standard in shape and that the fluctuations of this random function can be attributed only to the total area of the current, that is to the total charge released at the output.

Therefore, for multiplication factors sufficiently large the random function $J(t|x)$ current pulse at the output, can be written as:

$$J(t|x) = A\, i(t|x)$$

where $i(t|x)$ is the mean output current pulse and A is a statistical variable whose mean is one ($\bar{A} = 1$) and whose variance is: $\text{Var}\, A = \varepsilon^2(x)/M^2(x)$. The noise zero frequency intensity $s(0|x)$ depends strongly on the ratio β/α in the range $0 \leq \beta/\alpha \leq 1$ and is, for fixed multiplication factors $M(0)$, an increasing function of β/α. In fig. 15 it is shown the normalized noise intensity $s(0|0)/2FM^2(0)$ as a function of β/α for various values of $M(0)$.

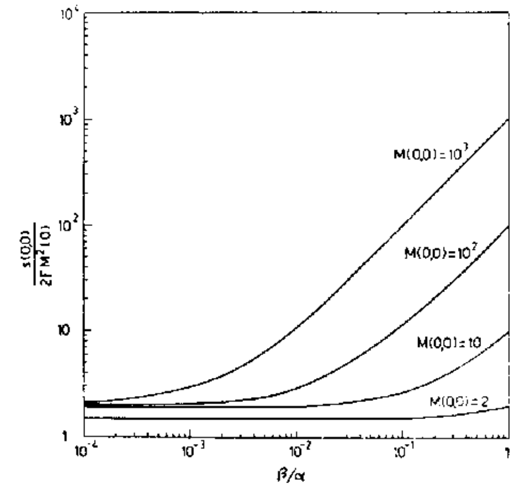

Fig. 15. — The normalized noise intensity of the avalanche diode excited at $x = 0$ by a pair for various $M(0)$.

As far as the influence of the scattering velocities v_e and v_h is concerned, we have seen that as far as the cutoff frequency of mean response and noise power spectrum is concerned, the only interesting velocity parameter is the effective scattering velocity $v^* = v_e v_h/(v_e + v_h)$. We have already seen (cfr. eqs. (29), (30), (43) and (44)) that this result holds in the approximate analytical relations; moreover we have tested its validity in different interesting cases with numerical computations.

APPENDIX

Let us consider the behaviour of the avalanching photodiode when excited, at a generic point x, by a number FT of pairs each having an independent p.d.f.

$$\frac{1}{T}e^{-\tau/T}$$

for the time of occurrence. Eq. (8′) reads in this case.

$$\Phi_*(\chi_1, t_1; \chi_2, t_2|x)$$

$$= \left\{\int_0^\infty \Phi(\chi_1, t_1 - \tau; \chi_2, t_2 - \tau|x)\,\frac{e^{-\tau/T}}{T}\,d\tau\right\}^{FT}$$

and, according to eqs. (9) and (10) the mean and the autocorrelation function can readily obtained:

$$q_*(t_1|x) = F\int_0^\infty q(t_1 - \tau|x)\,e^{-\tau/T}\,d\tau$$

$$\Gamma_*(t_1, t_2|x) = F\left(F - \frac{1}{T}\right)\cdot$$

$$\cdot \int_0^\infty q\,(t_1 - \tau|x)\, e^{-\tau/T}\, d\tau \int_0^\infty q\,(t_2 - \theta|x)\, e^{-\theta/T}\, d\theta\, -$$

$$+ F \int_0^\infty \Gamma\,(t_1 - \tau;\, t_2 - \tau|x)\, e^{-\tau/T}\, d\tau\,.$$

Again, one can at once obtain the mean and the autocorrelation function of the output current by derivation respect to t_1 and t_2:

(A1) $$i_*\,(t_1|x) = F \int_0^\infty i\,(t_1 - \tau|x)\, e^{-\tau/T}\, d\tau$$

(A2) $$\Lambda_*\,(t_1, t_2|x) = F\left(F - \frac{1}{T}\right) \cdot$$

$$\cdot \int_0^\infty i\,(t_1 - \tau|x)\, e^{-\tau/T}\, d\tau \int_0^\infty i\,(t_2 - \theta|x)\, e^{-\theta/T}\, d\theta\, +$$

$$+ F \int_0^\infty \Lambda\,(t_1 - \tau;\, t_2 - \tau|x)\, e^{-\tau/T}\, d\tau\,.$$

Letting $T \to \infty$, being Poisson-distributed the number of events in an interval of fixed length, one obtain mean $i_*\,(t_1/x)$ and autocorrelation function $\Lambda_*\,(t_1, t_2|x)$ of the output current which is stationary for times t_1 and t_2 sufficiently large to die out the initial transient; so from the following eqs.:

$$i_*\,(t_1|x) = F \int_0^\infty i\,(t_1 - \tau|x)\, d\tau$$

$$\Lambda_*\,(t_1, t_2|x) = F^2 \int_0^\infty i\,(t_1 - \tau|x)\, d\tau \int_0^\infty i\,(t_2 - \theta|x)\, d\theta\, +$$

$$+ F \int_0^\infty \Lambda\,(t_1 - \tau;\, t_2 - \tau|x)\, d\tau$$

we obtain for the stationary case, that is, for t_1 and t_2 very large, being

$$\lim_{t_1 \to \infty} \int_0^\infty i\,(t_1 - \tau|x)\, d\tau = M\,(x):$$

(A3) $$i_*\,(t_1|x) = F\, M\,(x)$$

(A4)

$$\Lambda_*\,(t_1, t_2|x) = F^2\, M^2\,(x) + F \int_0^\infty \Lambda\,(\theta, \theta + t_2 - t_1|x)\, d\theta\,.$$

Therefore, by the definition of spectral density:

(A5) $$s\,(\omega|x) = 2 \int_0^\infty e^{-j\omega\tau}\, K_*\,(\tau|x)\, d\tau$$

where

(A6) $$K_*\,(t_2 - t_1|x) = \Lambda_*\,(t_2 - t_1|x) - F^2\, M^2\,(x)$$

is the autocovariance function of the stationary random output current, letting

$$\Lambda\,(p_1, p_2|x) = \int_0^\infty \int_0^\infty \Lambda\,(t_1, t_2|x)\, e^{-p_1 t_1 - p_2 t_2}\, dt_1\, dt_2$$

we obtain, taking into account eqs. (A4), (A5) and (A6):

(A7) $$s\,(\omega|x) = 2\, F\, \Phi\,(j\,\omega, -j\,\omega|x)\,.$$

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BIBLIOGRAPHY

[1] H. W. Ruegg: « IEEE Trans. on El. Dev. », vol. E.D. 14, May 1967, 5.

[2] I. Ogawa: « Nucl. Instr. and Meth. », 49, 1967, 325.

[3] R. J. McIntyre: « IEEE Trans. on El. Dev. », vol. ED-13, n. 1, Jan. 1966.

[4] J. J. Chang: « IEEE Trans. on El. Dev. », vol. ED-14, n. 3 March 1967.

[5] R. B. Emmons and G. Lucovsky: « IEEE Trans. on El. Dev. », vol. ED.13, n. 3, March 1967, 297.

[6] R. B. Emmons: « J. of Appl. Phys. », 38, 9, August 1967, 3705.

[7] M. E. Hines: « IEEE Trans. on El. Dev. », vol. E.D.13, n. 1, Jan. 1966.

[8] S. Osnaghi: « Energia Nucleare », 11, 1967, 643.