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Optimum Signal Processing for Distance Measurement with Lasers

E. Gatti and S. Donati

The measurement accuracy of laser telemeters have been analyzed, taking into account the statistical properties of the propagation, detection, and measure process. For an arbitrary waveform of the light intensity modulation envelope, the optimum filter response and the resulting accuracy are found. As special cases, the optical radar and the sine wave modulation techniques have been considered, and their accuracy and optimum mode of operation are evaluated and discussed.

I. Introduction

Distance measurements with optical techniques have fully exploited the inherent properties of laser sources and greatly extended their range and precision capabilities. For example, the high degree of coherence of frequency stabilized lasers presently allows operation of interferometric instruments[1] over ranges of several hundred meters, with a limitation set by the medium degradation of coherence rather than by the laser source.

An important class of optical telemeters for the measurement of large distances is based on the intensity modulation of a laser beam. Two modes of operation have been developed independently: the sine wave mode (SWM)[1-3] and the pulsed mode (PM), also called the optical radar.[1,4,5] Both techniques use the laser as the source because of its high power and low divergence (i.e., high brillance), while monocromaticity, although not necessary in principle, is utilized to obtain discrimination against background light at the detector.

Basically, in the SWM the source can be either a cw laser operating in connection with an external electrooptical modulator, properly driven to obtain a sine wave intensity modulation, or a mode locked laser.[1] (In this case the fundamental component of the modulation envelope is selected after propagation and detection.)

The time delay of light returning to the receiving device after the propagation is measured from the phase shift of the modulation waveform, as usually obtained by heterodyning the electrical signal from the receiver detector. Since phases are measured mod. 2π, the distance is given apart from an integer number of modulation wavelengths, a feature similar to that of interferometric methods; however, in the SWM this number can be recovered in several ways, for example, by chirping the modulation waveform[1] or by combining the results obtained with different modulation frequencies.[1,6]

In the PM optical telemeter,[1,4,5] the source is a Q-switched laser, a PTM laser,[7] or a Q-switched mode-locked laser with single pulse extraction. The propagation time of the light pulse to the retroreflector and back to the receiver is measured by means of a fast, delayed time analyzer. The use of a high peak power laser in connection with a sensitive receiver, such as a photomultiplier coupled to a telescope, allows operation of the PM telemeter over very long distances, the most sound example being the well known Apollo 11 laser ranging experiment.[5]

To compare PM and SWM techniques, a detailed analysis of the measurement accuracy is made, starting from a statistical approach of the problem that allows one to take into account the random properties of propagation, detection, and measure processes. Then, the optimum treatment of information is found. As special cases, the optimum results for the measurement accuracy of the PM and SWM are calculated and compared.

II. Statistical Analysis

Let us denote with $As(t)$ the intensity of light leaving the source, where A is the average transmitted power and $s(t)$ is the intensity modulation waveform. After the propagation to the retroreflector and back for a distance $2L$, the light intensity is attenuated by a factor $\beta \exp[-2\alpha(t)L]$, $\alpha(t)$ being the random function that describes the medium attenuation and β being the collection efficiency of optics along the propagation path. The intensity $I(t)$ of light at the detector input is then given by

The authors are with CISE, P.O. Box 3986, 20100 Milano, Italy.

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$$I(t) = As(t)\beta \exp[-2\alpha(t)L],$$

and the mean value $\bar{I}(t)$ is therefore

$$\bar{I}(t) = As(t)\beta\, \overline{\exp[-2\alpha(t)L]}.$$

By introducing the fractional fluctuation $f(t)$ of the medium attenuation,

$$f(t) = \exp[-2\alpha(t)L] / \overline{\exp[-2\alpha(t)L]} - 1, \tag{1}$$

Eq. (1) can be written as

$$I(t) = \bar{I}(t)[1 + f(t)]. \tag{2}$$

After a photodetection with a quantum efficiency η, one has a received signal $S_r(t)$ with a mean value $\bar{S}_r(t)$ given by

$$\bar{S}_r(t) = \eta \bar{I}(t). \tag{3}$$

The fluctuations $\Delta S_r(t)$ from the mean value $\bar{S}_r(t)$ are due to the following mutually independent contributions: (1) attenuation coefficient fluctuations, which give a contribution $\eta[I(t) - \bar{I}(t)] = \eta\bar{I}(t)f(t) = \bar{S}_r(t)f(t)$ [see Eqs. (2) and (3)]; (2) the shot noise $R(t)$ of detected photoelectrons, which is associated with the instantaneous detected signal $\eta I(t) = \bar{S}_r(t)[1 + f(t)]$; (3) the signal-independent noise $Q(t)$ at the receiver, which includes background noise, dark current noise of the photodetector, and electrical noise of associated circuitry. The noise $Q(t)$ can be reported to the detector input and given in term of an equivalent rate ρ of dark photoelectrons.

The fluctuation $\Delta S_r(t)$ of the received signal can then be written as

$$\Delta S_r(t) = \bar{S}_r(t)f(t) + R(t) + Q(t). \tag{4}$$

Now, let us characterize the random properties of the received signal by the covariance function so defined[8,9]:

$$K_r(t,t') = \overline{\Delta S_r(t)\Delta S_r(t')}. \tag{5}$$

By inserting Eq. (4) in Eq. (5) and taking into account that f, R, and Q are uncorrelated random functions, we obtain

$$K_r(t,t') = \overline{f(t)f(t')}\bar{S}_r(t)\bar{S}_r(t') + \overline{R(t)R(t')} + \overline{Q(t)Q(t')}. \tag{6}$$

The signal-independent noise $Q(t)$ can be assumed as a white noise[8,9] with an intensity equal to the mean rate ρ of dark photoelectrons, i.e., $\overline{Q(t)Q(t')} = \rho\delta(t' - t)$. The shot noise $R(t)$, considered as a random function, can be written as the product $R_0(t)$ $[1 + f(t)]$ of the shot noise $R_0(t)$ associated with the mean received signal $\bar{S}_r(t)$ and of the medium attenuation random function $1 + f(t)$. Since the intensity of the white noise $R_0(t)$ is $\bar{S}_r(t)$ and being $\overline{f(t)} = 0$, one can easily find $\overline{R(t)R(t')} = \bar{S}_r(t)\delta(t' - t)[1 + \overline{f(t)f(t')}]$.

Denoting with $K_f(t,t') = \overline{f(t)f(t')}$ the covariance of fractional attenuation fluctuations and by substitution in Eq. (6), we obtain

$$K_r(t,t') = K_f(t,t')[\bar{S}_r(t)\bar{S}_r(t') + \bar{S}_r(t)\delta(t' - t)] + [\bar{S}_r(t) + \rho]\delta(t' - t). \tag{7}$$

The detected signal of which Eqs. (3) and (7) give the mean value and covariance can now be considered as the input signal to the photodetector and the linear shaping filter cascaded to it.

If $h(t)$ is the impulse response of such a cascade, the output random signal has a mean value $\bar{S}_h(t)$ and a covariance $K_h(t,t')$ that are given by the convolution theorems[8,9]:

$$\bar{S}_h(t) = \int_{-\infty}^{\infty} \bar{S}_r(t - \tau)h(\tau)d\tau, \tag{8}$$

$$K_h(t,t') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_r(t - \tau, t' - \tau')h(\tau)h(\tau')d\tau d\tau'. \tag{9}$$

For $t = t'$, Eq. (9) yields the amplitude variance $\epsilon_s^2(t)$ [by the Eq. (5)] of the output random signal $S_h(t)$ at time t. This is a datum about the amplitude information contained in the output signal at time t. Similarly, the time information is provided by the variance $\epsilon_\tau^2(t)$ of a threshold crossing time of the output signal if the threshold is located at the mean amplitude $\bar{S}_h(t)$ of the output signal at time t.

One can obtain the time variance $\epsilon_\tau^2(t)$ simply as the ratio between the amplitude variance $\epsilon_s^2(t)$ and the square of signal mean slope[10,11]

$$\epsilon_\tau^2(t) = K_h(t,t)/[d\bar{S}_h(t)/dt]^2. \tag{10}$$

Equation (10) strictly applies only for a linear random process, i.e., whether the slope of the random signal is constant in the threshold crossing interval.[10,11] The condition is satisfied if the practical requirement of a small measure error $\epsilon_\tau(t)$ is met.

At this point we can look for the optimum linear filter that, cascaded to the photodetector, gives an output signal whose threshold crossing at a suitable level yields all the timing information contained in the random detected signal. The impulse response of such an optimum filter can also be considered as the optimum weight with which the different part of the detected signal must be brought together at the measure time to recover the timing information through a threshold crossing. This measure method may appear less wieldy for the SWM than for the OR, but it will be shown later that the results directly suggest the optimum measure procedure in both cases. To find the optimum filter, let us look for the functional minimum of the time variance $\epsilon_\tau^2(t)$, given by Eq. (10), with respect to the filter impulse response $h_f(t)$. Let T_m be the total time duration of $h_f(t)$, obviously chosen so as to be larger or at least equal to the signal time duration; the variance $\epsilon_\tau^2(t)$ can then be minimized at the time T_m. The threshold amplitude need not be specified, since it will be provided as a result.

By means of standard variational methods[12] [one assumes that $h(t) = h_0(t) + \alpha\eta(t)$ in Eq. (10) and then settles $\partial\epsilon_\tau^2(t)/\partial\alpha = 0$ for $\alpha = 0$, see Appendix A] it can be found that $h(t)$ must satisfy, in the general case, the following Fredholm's linear integral equation of the first kind,

$$\int_{-\infty}^{\infty} K_r(t,t')h(T_m - t')dt' = \frac{d}{dt}\bar{S}_r(t), \tag{11}$$

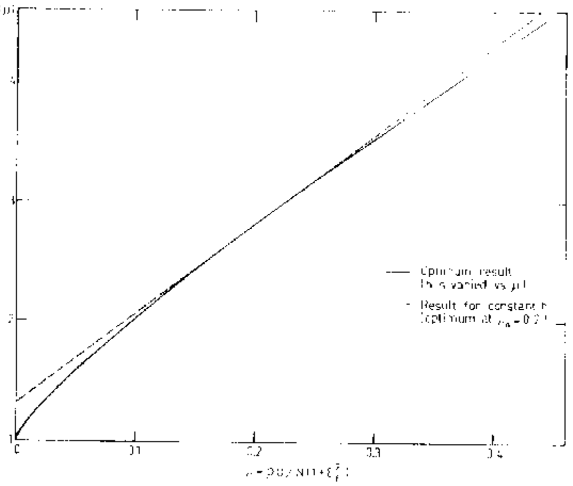

Fig. 1. The multiplicative factor $D(\mu)$ of the time variance in the PM telemeter against the dark noise parameter μ. Full line refers to the optimum result, i.e., the filter impulse response is varied with ρ according to Eq. (12). Dotted line represents the result obtained with the filter which is optimum at $\mu = 0.2$.

where T_m is the time at which the time variance becomes minimized.[13] Note that $h(T_m - t)$ is the time-specular waveform of the effective impulse response $h(t)$. The solution of Eq. (11) is straightforward if we assume that the attenuation fluctuations have a correlation time long if compared to signal duration or period (frozen propagation path), so that $K_f(t, t')$ has a constant value ϵ_f^2 equal to the fractional attenuation variance. This condition is satisfied in practice since there is a ratio of many orders of magnitude between the characteristic frequencies of attenuation fluctuations and the commonly used values of modulation frequency in the SWM or of reciprocal pulse duration in the PM. In this case, by substitution of Eq. (7) in Eq. (11) the optimum filter impulse response $h(t)$ turns out to be

$$h(T_m - t) = [d\bar{S}_r(t)/dt]/[(1 + \epsilon_f^2)\bar{S}_r(t) + \rho]. \quad (12)$$

Accordingly, the mean output signal $\bar{S}_h(t)$ from the filter and the time variance $\epsilon_h^2(t)$ at the measure time T_m, using Eqs. (8)–(10), are found to be

$$\bar{S}_h(T_m) = 0, \quad [d\bar{S}_h(t)/dt]_{t=T_m} > 0, \quad (13)$$

$$\epsilon_h^2(T_m) = 1 \Big/ \int_{-\infty}^{+\infty} \frac{[d\bar{S}_r(t)/dt]^2}{(1 + \epsilon_f^2)\bar{S}_r(t) + \rho} dt. \quad (14)$$

Thus, the signal at the optimum filter output has a zero mean value at the measure time, i.e., the measure consists in a zero crossing detection. Obviously, $h(T_m - t)$ represents the optimum weight attributed to the random signal $S_r(t)$ at the measure time T_m. The relative weight is heavier the greater is the ratio between the mean slope and the noise intensity.

III. PM Optical Telemeter

To evaluate the measurement accuracy of the PM telemeter, the transmitted pulse waveform has to be chosen. A suitable choice seems to be a waveform described by a single time parameter directly related to the pulse rise time, such as a gaussian with a standard deviation σ.

If N is the mean total number of collected photoelectrons, the mean received pulse $\bar{S}_r(t)$ can be written as

$$\bar{S}_r(t) = \frac{N}{(2\pi\sigma)^{\frac{1}{2}}} \exp[(t - \tau_0)^2/2\sigma^2]. \quad (15)$$

By inserting Eq. (15) in Eq. (14), one can get for the optimum variance of the time measurement

$$\epsilon_h^2 = \frac{(1 + \epsilon_f^2)\sigma^2}{N} D(\mu), \quad (16)$$

where

$$D(\mu) = 1 \Big/ \int_{-\infty}^{+\infty} \frac{[\xi^2/(2\pi)^{\frac{1}{2}}] \exp(-\xi^2/2)}{1 + \mu(2\pi)^{\frac{1}{2}} \exp(\xi^2/2)} d\xi \quad (17)$$

and

$$\mu = \rho\sigma/N(1 + \epsilon_f^2). \quad (18)$$

The factor $D(\mu)$ represents the effect due to the dark photoelectrons noise ρ, which is taken into account through the parameter μ. For $\rho = 0$, one has $\mu = 0$ from Eq. (18), and Eq. (17) gives $D(0) = 1$. Therefore, the first term at the right-hand side of Eq. (16) gives the effect of shot noise and attenuation fluctuations alone. In the general case, the function $D(\mu)$ has been calculated by means of numerical integration of Eq. (17), and the results are reported in Fig. 1 vs the noise-to-signal parameter μ. The dependence of the optimum filter impulse response upon μ can be seen from the diagrams of Fig. 2. When μ is very small, the optimum weight that is attributed to the signal is proportional to $(t - t_0)/\sigma$, i.e., the filter yields the centroid position as the best defined time localization of the random pulse. The more μ increases, the shorter becomes the time duration of the optimum filter impulse response, and the point of maximum weight is progressively shifted toward that of maximum signal slope.

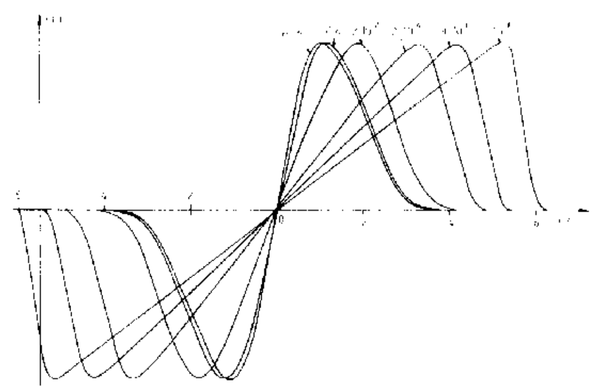

Fig. 2. The waveforms of the optimum filter impulse response $h(t)$ for different values of the dark noise parameter μ (amplitudes not to the same scale) in a PM telemeter.

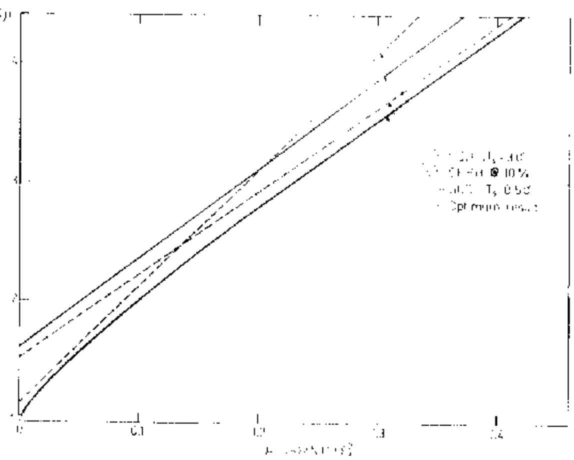

Fig. 3. The effect of simple approximations of the optimum filter impulse response. The multiplicative factor $D(\mu)$ of Eq. (16) is plotted for a short circuited delay line shaping (SCDL) with a shaping time T_s (dotted line) and for a constant fraction pulse height (CFPH) at the 10% amplitude level (full line) together with the optimum result.

For $\mu \approx 0.2$ or greater, the optimum filter response is very close to a gaussian derivative. Now, suppose that the optimum filter for the value $\rho = \rho_0$ of the dark photoelectron noise is still used when $\rho \neq \rho_0$. Then, it can be shown (see Appendix B) that the diagram of the time variance vs ρ (or μ) is simply the tangent straight line to the curve of Fig. 1 at the value μ_0 corresponding to ρ_0. As an example, the dotted line in Fig. 1 represents the case $\mu_0 = 0.2$. Note that the difference with the absolute optimum results is not very large for a fairly wide range of μ around μ_0. Once the optimum filter response for a value $\mu = \mu_0$ has been selected, the term $D(\mu)$ can be simply written as linearly proportional to μ,

$$D(\mu) = D(\mu_0) + (\mu - \mu_0)D'(\mu_0), \tag{19}$$

and therefore Eq. (16) reduces to

$$\epsilon_t^2 = [(1 + \epsilon_f^2)\sigma^2/N]D_0 + (\rho\sigma^3/N^2)D'(\mu_0), \tag{20}$$

where $D_0 = D(\mu_0) - \mu_0 D'(\mu_0)$ is slightly greater than unity, while $D'(\mu_0)$ is of the order of 10, in the range $0.01 < \mu < 1$.

The dark noise contribution to the normalized time variance ϵ_t^2/σ^2 is proportional to the ratio between the average number of dark photoelectrons falling in the measure interval, $\rho\sigma$, and the squared average number of signal photoelectron N^2. It is now interesting to compare the optimum results to those obtained by relatively simple timing networks. Let us assume that the detector has an impulse response equal to the transmitted pulse waveform, i.e., a gaussian pulse with a width σ. Then, a simple approximation of the optimum filter response can be obtained by shaping the detector output with a short circuited delay line (SCDL) or by the constant fraction pulse height (CFPH)[14] method of zero crossing. The time variance for these two illustrative cases can still be written as in Eq. (16), and the comparison is therefore reduced to that of the $D(\mu)$ multiplicative factors. These have been computed by numerical integrations of Eqs. (7) (10), and the results are reported in Fig. 3. Note that the optimum is fairly well approached. Apparently, the SCDL is superior to the CFPH, but this is only due to the particular shape of the transmitted pulse. If the case of a Q-switched laser pulse with nearly exponential leading edge and slower trailing edge were considered, the converse applies, since here the CPFH correctly would attribute the major weight to the faster part of the signal. A critical consideration about the validity of Eq. (16) has now to be made. Usually, the time $2L/c$ is measured as the difference between the zero crossing time of the start pulse (obtained by returning to the receiver a small fraction of the optical pulse leaving the transmitter) and the stop pulse due to the light returning after propagation. Equation (16) only gives the accuracy of timing and does not take into account that the measure channel can be stopped by a dark photoelectrons pulse. The problem is of no importance if the mean number N of photoelectrons of the received signal is large, since in this case an amplitude discrimination suffices to obtain discrimination of dark pulses.

Further effects may give a substantial contribution to the time variance. For example, the detector time jitter introduces an error that obviously cannot be reduced by optimum filtering. If ϵ_j^2 is the variance of the time jitter, the contribution to the measure variance is ϵ_j^2/N. If a photomultiplier is used, the photocathode-first dynode time of flight is the most important source of jitter,[10] since the effect of dynodes time of flight is smoothed by the large number of electrons. Also, a systematic component might arise if received light is not uniformly spread over the photocathode surface from measure to measure.

Another effect of importance can be the lack of uniform time distribution in the emitted light beam. If the source is a Q-switched laser[15] in which the various modes do not build up with the same gain or start at the same time, the light pulse leaving the transmitter has a waveform that results from the superposition of many modes waveforms. Because of their different directions, the modes will be in general unevenly attenuated at the retroreflector (or at the receiver), and the weights with which they are superposed at the receiver will be no more proportional to those giving the transmitted pulse. This effect causes the mean waveform of transmitted and received pulses to be different, and a systematic error in timing is introduced. The error cannot be removed simply by a calibration, since it is distance dependent. Moreover, atmospheric turbulence enhances the effect further through the random spreading and steering of the laser beam and also provides an additional fluctuation of the received pulse waveform.

Even if the error introduced is relatively small as compared to σ, its contribution to the time variance becomes the most important once the statistical errors of Eq. (20) are smoothed down by averaging on very large numbers of photoelectrons.

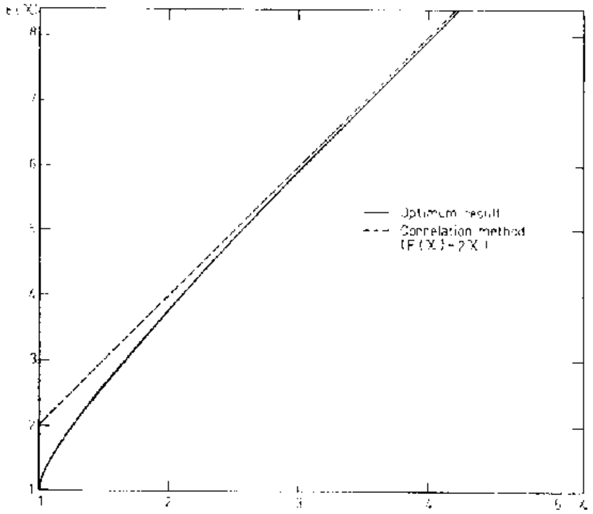

Fig. 4. The multiplicative factor $E(\chi)$ of the time variance in the SWM telemeter [Eq. (24)] plotted against the dark noise parameter χ [Eq. (22)]. Dotted line refers to the result obtained with the correlation method of measure.

IV. SWM Optical Telemeter

In this technique the transmitted beam has an intensity waveform that can be written as

$$s(t) = 1 + m \sin\Omega t,$$

m being the modulation index and Ω the modulation frequency. Therefore, the mean received signal is given by

$$\bar{S}_r(t) = (N/T)[1 + m \sin\Omega(t - \tau_0)],$$

where N is the mean total number of signal photoelectrons collected in the measure interval lasting a total time T. By applying Eq. (12), the optimum filter impulse response is found to be

$$h(t) \propto \frac{\cos\Omega(T_m - t),}{\chi + \sin\Omega(T_m - t)}, \qquad (21)$$

where

$$\chi = (1/m)\{1 + [\rho T/(1 + \epsilon_f^2)N]\}. \qquad (22)$$

Accordingly, the time variance of the measurement can be obtained from Eq. (14). Since the measure lasts a total time T, the upper limit of the integral must be replaced with T. Noting also that the integrand is periodic, the integral is given by $T/(2\pi/\Omega)$ times the value of a single period. In this way we can obtain

$$\epsilon_h^2 = [(1 + \epsilon_f^2)/m\Omega^2 N]E(\chi), \qquad (23)$$

where

$$E(\chi) = 2\pi \Big/ \int_0^{2\pi} \frac{\cos^2\xi}{\chi + \sin\xi}\, d\xi. \qquad (24)$$

The noise parameter χ is always greater than 1, as seen from Eq. (22). If the noise is strong enough to mask the signal, $\chi \gg 1$, and the optimum filter is $h(t) = \cos\Omega \times (T_m - t)$, i.e., a cosinusoid at the signal frequency. The optimum filter yields in this case a phase-sensitive signal

$$\int_0^{T_m} S_r(t) \cos\Omega t dt,$$

which crosses the zero level at the measure time T_m. Note that the same signal is obtained by correlating the received signal with a reference sine wave in the technique used to implement the SWM telemeter. From Eq. (24) one has $E(\chi) = 2\chi$ for $\chi \gg 1$, and therefore the time variance ϵ_h^2 becomes

$$\epsilon_h^2 = \frac{1 + \epsilon_f^2}{\Omega^2(m^2/2)N} + \frac{\rho T}{\Omega^2(m^2/2)N^2}. \qquad (25)$$

Now, give to $1/\Omega^2$ the same scale factor meaning as that of σ^2 in the PM technique. Then, the shot noise and attenuation fluctuations contributions are readily seen to be almost the same in the two cases (except for $m^2/2$, a modulation efficiency factor). Also, the dark noise contribution is still given by the noise (ρT) to the squared signal (N^2) ratio of the average number of photoelectrons in the measure interval. Nevertheless, this ratio is quantitatively much larger than that of the PM technique because of the order-of-magnitudes difference in the signal time durations T and σ.

In the general case, the multiplicative factor $E(\chi)$ has been computed by means of numerical evaluation of Eq. (24), and the results are plotted in Fig. 4. The corresponding impulse response of the optimum filter is reported in Fig. 5 for some values of χ. Again, the result obtained with the correlation method of measure is given by the tangent straight line to the $E(\chi)$ diagram for $\chi \gg 1$. As seen from Fig. 4, $E(\chi) = 2\chi$ in this case, and Eq. (25) gives not only the optimum result for $\chi \gg 1$ but also the variance obtained with the correlation method for any value of χ (or ρ). The loss with respect to the optimum result is within a factor of 2, which is reached only for $\chi \approx 1$, i.e., when the dark noise ρ is very small, and the modulation index m is very near to unity. The optimum filter response, which is a

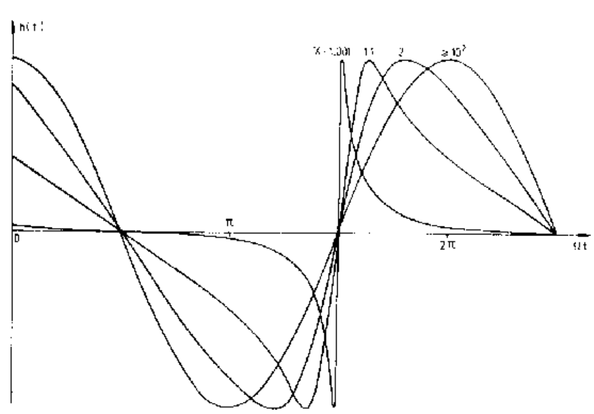

Fig. 5. The optimum filter response $h(t)$ for some values of χ. Since $h(t)$ is periodic, the diagram has been limited to one period. Amplitudes are not to the same scale.

cosinusoid for $\chi \gg 1$, appreciably changes its shape only for $\chi \approx 1$ and at last becomes an approximate succession of $\delta(t)$ derivatives (see Fig. 5), suggesting that the best timing is in this case provided by the periodic derivation of the signal.

The counterpart of the effects discussed in Sec. III for the OR must be taken into account also in the SWM operation. Especially important is the time jitter of photocathode-first dynode time of flight if a photomultiplier is used as the detector. This is equivalent to a phase jitter on the output signal and may give a further contribution to the time variance ϵ_τ^2.

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Appendix A

From Eqs. (8)–(10), the time variance $\epsilon_\tau^2\ (T_m)$ at the time $t = T_m$ can be written, with self-explanatory notations, in the following way:

$$\epsilon_\tau^2(T_m) = \{[K_\tau(t,t')\overset{*}{*}h(t)h(t')]/[\dot{S}_\tau(t)*h(t)]^2\}_{t=t'=T_m}.$$

Letting $h(t) = h_0(t) + \alpha\eta(t)$, where $\eta(t)$ is an arbitrary function and, taking $\partial\epsilon_\tau^2\ (t)/\partial\alpha = 0$ at $\alpha = 0$ yields after simple rearrangements the following condition:

$$\left[\frac{K_\tau(t,t')\overset{*}{*}\eta(t)h_0(t')}{K_\tau(t,t')\overset{*}{*}h_0(t)h_0(t')}\right]_{t=t'=T_m} = \left[\frac{\dot{S}_\tau(t)*\eta(t)}{\dot{S}_\tau(t)*h_0(t)}\right]_{t=T_m}.$$

This condition is satisfied at $t = T_m$ and for any $\eta(t)$ by assuming $h(t)$ such that

$$[K_\tau(t,t')*h_0(t')]_{t'=T_m} = \dot{S}_\tau(t),$$

which is coincident with Eq. (11) except for the notations.

Appendix B

From Eqs. (8)–(10), it can be seen that the time variance is a linear function of the dark noise ρ, therefore the diagram of $D(\mu)$ for a specified filter $h(t)$ is a straight line. Moreover, if $h(t)$ is the optimum filter for the value $\mu = \mu_0$, the straight line can only be the tangent to the optimum $D(\mu)$ diagram in the point $\mu = \mu_0$. The negative curvature of this latter comes as a consequence, since all the tangent straight lines must necessarily lie above the optimum curve.

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