# Speckle-pattern intensity and phase: Second-order conditional statistics

**Silvano Donati and Giuseppe Martini**
*Istituto di Elettronica, Università di Pavia, 27100 Pavia, Italy*
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An analysis of the second-order conditional statistics of speckle patterns is developed, under the assumption of Gaussian field components. After deriving the conditional distributions and moments of intensity and phase for a joint measurement performed in the presence of partial correlation, the results of joint and conditional phase-variances are applied to determine the accuracy limits of speckle-pattern interferometric measurements.

## INTRODUCTION

Coherence properties of speckle patterns, which are the starting point of various instrumental methods, can be analyzed by means of a second-order statistical approach. A basic treatment is supplied by the well-known works of Goodman,[1,2] who derived the joint distribution and moments of speckle intensity and phase. In this paper, we analyze the second-order *conditional* probabilities and moments of the speckle, under the assumption of circular Gaussian statistics of the field components, i.e., for a fully developed, polarized speckle pattern produced by an ideal diffuser.

The conditional approach provides a statistical description of the speckle-pattern subsets, defined as those regions of trial space with the same specified value of a random variable. This description is not directly supplied by the usual joint statistics. For example, consider the statistics of the random phase $\vartheta$ of a speckle field, and its connection to an interferometric measurement of path difference $s_2 - s_1 = (\vartheta_2 - \vartheta_1)\lambda/2\pi$. First-order statistics gives no information, for it states only that, at a particular point, the phase is uniformly distributed on $-\pi,\pi$ and the phase variance is $(2\pi)^2/12$; the second-order joint statistics of the phases $\vartheta_1$ and $\vartheta_2$ accounts for nonzero correlation at two distances $z_1$ and $z_2$, and allows us to calculate the variance of $\vartheta_2 - \vartheta_1$ regardless of the intensities, i.e., on the whole speckle set; the conditional approach also includes the detailed dependence of $\vartheta_2 - \vartheta_1$ on the measured intensities (or amplitudes) at $z_2$ and $z_1$, which can be used, e.g., to select those speckle subsets that yield a phase variance smaller than the average.

Following the work of Middleton[3] as proposed by Goodman,[2,4] we treat the speckle as a narrow-band Gaussian noise. After deriving the conditional statistics of intensity and phase, we discuss the results with reference to laser-interferometry measurements of displacement.

## I. SPECKLE PATTERN CONDITIONAL PROBABILITIES AND MOMENTS

The joint probability of intensity and phase at two points $P_1$ and $P_2$ of a speckle pattern, whose field components obey a circular Gaussian distribution with zero mean value and variance $\sigma^2$, is the well-known four-dimensional density[2,4]:

$$p(I_1,I_2,\vartheta_1,\vartheta_2) = \frac{1}{16\pi^2\sigma^4(1-\mu^2)} \times \exp\left(-\frac{I_1 + I_2 - 2\sqrt{I_1 I_2}\,\mu\cos(\vartheta_1 - \vartheta_2 + \varphi)}{2\sigma^2(1-\mu^2)}\right), \quad (1)$$

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where $\mu_c = \mu \exp i\varphi$ is the complex coherence factor of the scattered field at points $P_1$ and $P_2$. From Eq. (1) the joint probability density of intensities or of phases alone is obtained by a double integration on the omitted variables. The result, which will be used in the following, is explicitly given by[2,3]

$$p(I_1,I_2) = \frac{1}{4\sigma^4(1-\mu^2)} \times \exp\left(-\frac{I_1+I_2}{2\sigma^2(1-\mu^2)}\right) \mathscr{I}_0\left(\frac{\mu\sqrt{I_1 I_2}}{\sigma^2(1-\mu^2)}\right), \quad (2)$$

where $\mathscr{I}_0$ is the modified Bessel function of first kind, zero order, and

$$p(\vartheta_1,\vartheta_2) = \frac{1-\mu^2}{4\pi^2} \frac{\sqrt{1-\beta^2} + \beta \arcsin\beta + (\pi/2)\beta}{(1-\beta^2)^{3/2}}, \quad (3)$$

in which $\beta = \mu\cos(\vartheta_1 - \vartheta_2 + \varphi)$.

Since Eqs. (1)–(3) cannot be separated for $\mu \neq 0$ in any form of probability product of the variables, none of these is statistically independent of the remaining ones. As a consequence, some information about intensity (or phase) which was contained in the phase (or intensity) variables of Eq. (1) has been lost in Eq. (2) [or Eq. (3)]. On the contrary, a conditional probability retains the information contained in all the displayed variables, either independent or conditioned. Thus, the accuracy of intensity (or phase) estimate can be improved for $\mu \neq 0$ over that indicated by Eq. (2) [or eq. (3)] by deriving from Eq. (1) a probability conditioned on the phase (or intensity). Obviously, the same remark applies to subensembles of the above four variables.

Let us begin with the probabilities of intensity $I_2$ conditioned on $I_1$, and of phase $\vartheta_2$ conditioned on $\vartheta_1$. By definition, we shall divide the joint probabilities of Eqs. (2) and (3) by the corresponding first-order densities $p(I_1) = (1/2\sigma^2)\exp(-I_1/2\sigma^2)$ and $p(\vartheta_1) = 1/2\pi$, and thus we find

$$p(I_2|I_1) = \frac{1}{2\sigma^2(1-\mu^2)} \times \exp\left(-\frac{I_2+\mu^2 I_1}{2\sigma^2(1-\mu^2)}\right) \mathscr{I}_0\left(\frac{\mu\sqrt{I_1 I_2}}{\sigma^2(1-\mu^2)}\right) \quad (4)$$

and

$$p(\vartheta_2|\vartheta_1) = p(\vartheta_2,\vartheta_1)2\pi. \quad (5)$$

It is now interesting to find the conditional mean values and variances of intensity and phase, and compare them with the "free" first-order values which are given by

$$\langle I_2\rangle = 2\sigma^2, \quad \sigma_{I_2}^2 = (2\sigma^2)^2,$$
$$\langle \vartheta_2\rangle = 0, \quad \sigma_{\vartheta_2}^2 = \pi^2/3. \quad (6)$$

From Eq. (4), the mean value of intensity $\langle I_2\rangle|_{I_1}$ conditioned on $I_1$ can be obtained as (using integral A.1.52 of Ref. 3)

$$\langle I_2\rangle|_{I_1} = 2\sigma^2(1-\mu^2) + \mu^2 I_1, \quad (7)$$

and for the variance $\sigma_{I_2}^2|_{I_1}$ of intensity $I_2$ conditioned on $I_1$ we have

$$\sigma_{I_2}^2|_{I_1} = [2\sigma^2(1-\mu^2)]^2 + 4\sigma^2\mu^2(1-\mu^2)I_1. \quad (8)$$

Note how both intensity mean and variance are even functions of the coherence factor $\mu$; their dependence on $|\mu|$ is shown in Fig. 1 for some values of the intensity $I_1$, with a normalization by $2\sigma^2$. As expected, the diagrams of normalized mean and variance start from unity for $\mu = 0$ and reach monotonically the full-correlation values $I_1/2\sigma^2$ and 0 for $|\mu| = 1$.

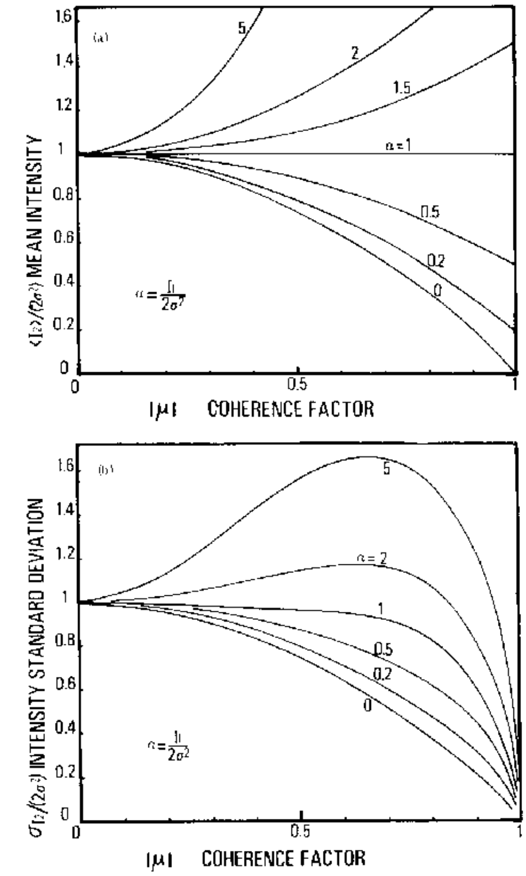


FIG. 1. (a) Mean value $\langle I_2\rangle$ and (b) standard deviation $\sigma_{I_2}$ of speckle pattern intensity $I_2$ conditioned on $I_1$ as a function of the coherence factor $\mu$.

Regarding the speckle phase density, we have from Eqs. (5) and (3) that the distribution depends on $\beta = \mu\cos(\vartheta_1 - \vartheta_2 + \varphi)$ and, therefore, it has an even symmetry with respect to both variables $\vartheta_1 - \vartheta_2 + \varphi$ and $\vartheta_1 - \vartheta_2 + \varphi + \pi$. The maximum of the distribution is at $\vartheta_1 - \vartheta_2 + \varphi = 0$ for $\mu > 0$ and at $\vartheta_1 - \vartheta_2 + \varphi = \pi$ for $\mu < 0$; in view of this symmetry the mean value of phase $\vartheta_2$ conditioned on $\vartheta_1$ is

$$\langle\vartheta_2\rangle|_{\vartheta_1} \begin{array}{ll} = \vartheta_1 + \varphi & (\mu > 0) \\ = \vartheta_1 + \varphi - \pi & (\mu < 0), \end{array} \quad (9)$$

and therefore the distribution of phase deviation $\vartheta_2 - \langle\vartheta_2\rangle$ is even and independent of the sign of $\mu$. Therefore, the variance $\sigma_{\vartheta_2}^2|_{\vartheta_1}$ of phase $\vartheta_2$ conditioned on $\vartheta_1$ can be written simply as $\int_{-\pi}^{\pi} \vartheta^2 p(\vartheta)\,d\vartheta$, where $p(\vartheta)$ is given by Eqs. (5) and (3) and $\vartheta = \vartheta_1 - \vartheta_2 + \varphi$. This integral is evaluated by parts with the help of the distribution function $P(\vartheta)$ associated with the density $p(\vartheta)$,

$$P(\vartheta) = \int_{-\pi}^{\vartheta} p(\vartheta')\,d\vartheta' = \frac{1}{2} + \frac{\vartheta}{2\pi} + \frac{\pi/2 + \arcsin\beta}{2\pi\sqrt{1-\beta^2}}\,\mu\sin\vartheta,$$

and the result is

$$\sigma_{\vartheta_2}^2|_{\vartheta_1} = \frac{\pi^2}{3} - \pi\arcsin|\mu| + \arcsin^2|\mu| - \frac{1}{2}\sum_{n=1}^{\infty}\frac{\mu^{2n}}{n^2}. \quad (10)$$

Since the series of the inverse squared integers is equal to $\pi^2/6$,

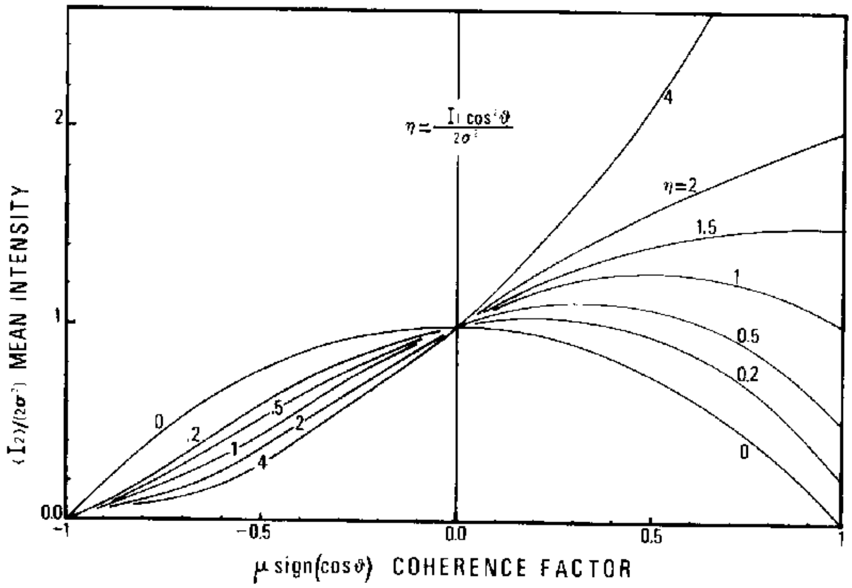


FIG. 2. Mean value of intensity $I_2$ conditioned on intensity $I_1$ and phase difference $\vartheta = \vartheta_1 - \vartheta_2 + \varphi$, as a function of the coherence factor. On the abscissa, $\mu$ is multiplied by the sign of $\cos\vartheta$ to allow for any sign combination of $\mu$ and $\cos\vartheta$.

Eq. (10) gives $\sigma_\vartheta^2 = 0$ for $|\mu| = 1$, and $\sigma_\vartheta^2 = \pi^2/3$ for $\mu = 0$ as expected. Equation (10) differs by a factor ½ from a result reported in Ref. 3, p. 411, Eq. 9.4.1 as the variance of the random variable $\vartheta_2 - \vartheta_1$, i.e., the phase difference. Actually, we have found with Eq. (10) the variance of $\vartheta_2$ conditioned on $\vartheta_1$, which obviously coincides with the variance of the variable $\vartheta_2 - \vartheta_1$ conditioned on $\vartheta_1$. For an interferometric measurement, the conditional variance is a better description of the accuracy than the free variance of $\vartheta_2 - \vartheta_1$, since $\vartheta_1$ bears no information in itself; this amounts precisely to half the variance as stated above.

By definition, the conditional mean values (7) and (9) represent the best estimates[5] in the quadratic sense of $I_2$ and $\vartheta_2$ at point $P_2$ based on the measurement of $I_1$ and $\vartheta_1$ at point $P_1$, while the conditional variances (8) and (10) are the quadratic errors of such estimates. This remark also applies to the moments derived below.

The most general second-order conditional probability of intensity $I_2$, as obtained from Eq. (1), is explicitly

$$p(I_2|I_1,\vartheta_1,\vartheta_2) = [2\sigma^2(1-\mu^2)]^{-1} \times \exp\left(-\frac{I_2 - 2\mu\sqrt{I_1 I_2}\cos\vartheta}{2\sigma^2(1-\mu^2)}\right) D(\delta), \quad (11)$$

where $\delta = \mu\cos\vartheta\sqrt{I_1/2\sigma^2(1-\mu^2)}$, $\vartheta = \vartheta_1 - \vartheta_2 + \varphi$, and

$$1/D(\delta) = 1 + \sqrt{\pi}\,\delta\exp(\delta^2)(1 + \mathrm{erf}\,\delta), \quad (12)$$

erf being the standard error function.

The mean value and variance of the conditional intensity then follow from Eq. (11), by using Eq. A.1.49 of Ref. 3, as

$$\langle I_2\rangle|_{I_1,\vartheta} = \sigma^2(1-\mu^2)[3 - D(\delta) + 2\delta^2], \quad (13)$$

$$\sigma^2_{I_2}|_{I_1,\vartheta} = \sigma^4(1-\mu^2)^2[6 - (1-2\delta^2)D(\delta) - D^2(\delta) + 8\delta^2], \quad (14)$$

and are plotted in Figs. 2 and 3 with $I_1\cos^2\vartheta/2\sigma^2$ as a parameter, and $\mu\,\mathrm{sign}(\cos\vartheta)$ as the abscissa to allow for any combination of signs of $\mu$ and $\cos\vartheta$. From a comparison with Fig. 1 we can see that the introduction of the phase as a condi-

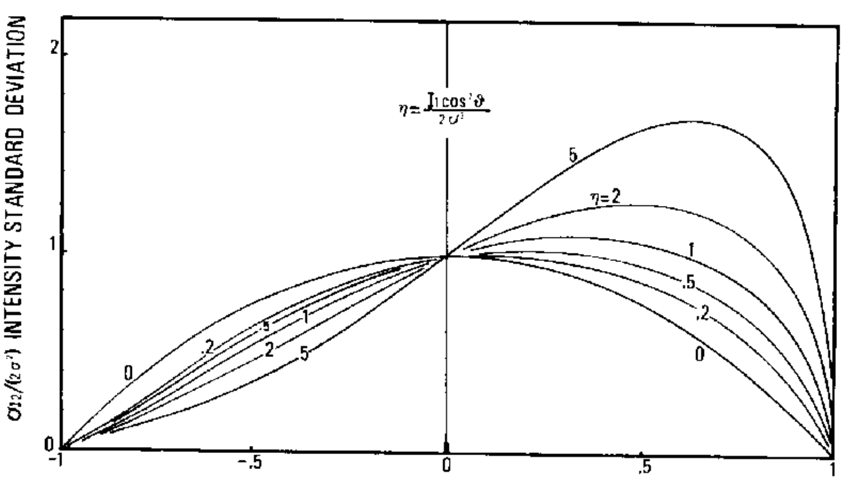


FIG. 3. Standard deviation of intensity $I_2$ conditioned on $I_1$ and $\vartheta$ versus the coherence factor.

tioned variable suppresses the symmetry with respect to $|\mu|$.

Similarly, the second-order conditional probability of phase $\vartheta_2$ is given by

$$p(\vartheta_2|I_1,I_2,\vartheta_1) = \frac{1}{4\pi^2}\exp\frac{\mu\sqrt{I_1I_2}\cos(\vartheta_1-\vartheta_2)}{\sigma^2(1-\mu^2)}\Big/\mathcal{I}_0\left[\frac{\mu\sqrt{I_1I_2}}{\sigma^2(1-\mu^2)}\right], \quad (15)$$

and because of the same symmetry with respect to $\vartheta_1 - \vartheta_2 + \varphi$ as that of Eq. (5), the mean value $\langle\vartheta_2\rangle$ is still given by Eq. (9), and the variance is dependent only on $|\mu|$. By expansion of the exponential term in Eq. (15), we can find for the variance,

$$\sigma^2_{\vartheta_2}|_{I_1,I_2,\vartheta_1} = \frac{\pi^2}{3} + \frac{4}{\mathcal{I}_0(z)}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\mathcal{I}_n(z), \quad (16)$$

where $z = |\mu|\sqrt{I_1I_2}/\sigma^2(1-\mu^2)$, and $\mathcal{I}_n$ is the $n$th order modified Bessel function of the first kind. In Fig. 4 the conditioned variance is plotted versus $|\mu|$, together with the "free" variance of phase given by Eq. (10). Note that, for a coherence factor near unity, the curve of the free variance intersects all those of conditional variances of decreasing parameter value. Thus, the contribution to the phase error is increasingly due to small-intensity speckles as $\mu \to 1$. This is due to the combined effect of quadratic composition of variances and of a weight $p(I_1,I_2)$ of the composition which increases for $\mu \to 1$ at small intensity values, as given by Eq. (2).

Equations (10) and (16) become indeterminate forms for $\mu \to 1$, and cannot be used directly in the region of high coherence. Letting $\mu = 1 - \zeta^2$, it is possible to obtain the following asymptotic behavior of the phase variances for small $\zeta$,

$$\sigma^2_\vartheta = \zeta^2(3 - \ln 2\zeta^2), \qquad \sigma^2_\vartheta|_{I_1I_2} = \zeta^2/(\sqrt{I_1I_2}/2\sigma^2). \quad (17)$$

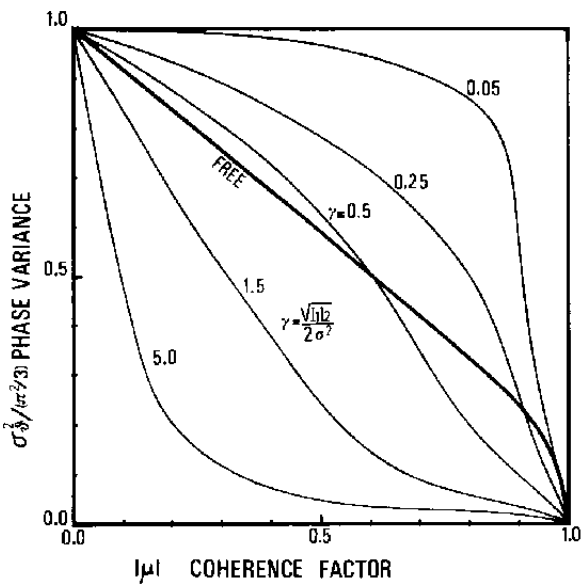


FIG. 4. Variance of the speckle phase $\vartheta_2$ conditioned on $\vartheta_1$. Thick line is for the free distribution (not conditioned on intensities); thin lines are for the variance conditioned also on intensities $I_1$ and $I_2$ entering as parameters.

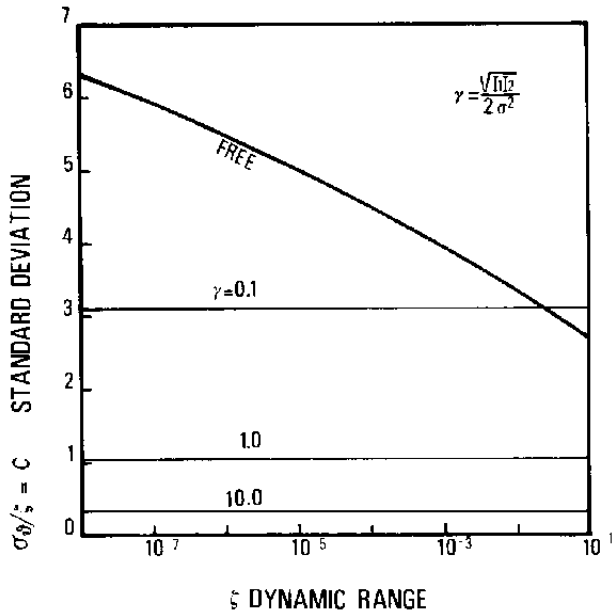


FIG. 5. Free and conditional phase standard deviation for coherence factors $\mu = 1 - \zeta^2$ near unity, plotted against $\zeta$.

Thus, the main dependence of both free and conditional variances is on $\zeta^2$, with a multiplicative factor that steadily increases as $\zeta \to 0$ in the former, and that is the inverse of speckle intensity $I \simeq \sqrt{I_1I_2}$ normalized to the mean value $2\sigma^2$ in the latter. The factor $C = \sigma_\vartheta/\zeta$ is plotted in Fig. 5.

## II. SPECKLE ERRORS IN INTERFEROMETRIC MEASUREMENTS

Laser interferometry is used to measure diffuse target displacements in a number of applications, including remote vibration measurement[6] in the visible and ir, surface acoustic wave detection,[7] and biological motility pickup.[8] Current performances are close to the quantum noise limit,[8] and noise equivalent displacements (NED) have been reported in the range $10^{-8}$ to $10^{-3}\lambda$, depending on detected power and signal frequency and bandwidth. At these sensitivity levels, the speckle statistics shall be considered as a possible source of error in the measurement of interferometric phase difference $\vartheta_2 - \vartheta_1 = \varphi$. Specifically, let us consider a laser interferometer transmitting a beam along the $z$ axis to a target placed in the $xy$ plane, so that a phase signal $\varphi = 2k(z_2 - z_1)$ is generated in the interferometer by the $z$ component of target displacement. Any effect changing the speckle projected back in the interferometer is the source of a phase error $\sigma_\vartheta$. This error can be compared with other sources of error by introducing a speckle NED defined as NED $= \sigma_\vartheta/2k$, where $k$ is the wave number. Clearly, a case of importance is that of small $\zeta$, i.e., of high coherence, for which $\sigma_\vartheta = C\zeta$ and, therefore, NED $= C\zeta/2k$. Thus, the factor $C$ reported in Fig. 5 shows how the speckle NED is affected by intensity in a conditional measurement.

Two main sources of speckle error are now briefly discussed, i.e.,

(i) Changes of the diffuser random sample illuminated by the laser spot, as due either to in-plane components of target displacement, or to beam deflections caused by turbulence in the propagation path; and,

(ii) optical path length changes, caused by slow fluctuations of the refraction index along the optical path or by large extraneous drifts of target distance, both changing the speckle region received in the interferometer. In absence of these effects, the signal itself changes the speckle at large amplitudes of displacement and limits the accuracy at a given dynamic range.

Following the procedure outlined by Goodman,[2] it is easy to obtain, for an ideal diffuser and in the Fresnel-Huygens approximation, the complex coherence factor $\mu_c$ associated with the above perturbations, when the beam is described by a Gaussian fundamental mode with spot size $w$ on the diffuser. For case (i), a lateral displacement $r$ of the spot with respect to the diffuser yields

$$\mu_c = \exp(-r^2/2w^2), \qquad (18)$$

where a correcting term, small for $z \gg kwr$, has been omitted. From Eq. (18), we have for $r \ll w$:

$$\mathrm{NED} = \frac{C}{2\sqrt{2}} \frac{r}{kw}. \qquad (19)$$

For case (ii), a displacement $z_2 - z_1$ of the diffuser along the beam axis gives

$$\mu_c = \frac{\exp\{2ik(z_2 - z_1) + i \arctan[k(z_2 - z_1)w^2/4z_2z_1]\}}{\{1 + [k(z_2 - z_1)w^2/4z_2z_1]^2\}^{1/2}}, \qquad (20)$$

where the first term in the exponential is the correct interferometric signal, the second is a correction due to wave-front curvature, and the denominator gives (for $\mu \approx 1$)

$$\mathrm{NED} = \frac{C}{8\sqrt{2}} \frac{(z_2 - z_1)w^2}{z_2z_1}. \qquad (21)$$

Finally, a comment is in order about the hypothesis of circular Gaussian statistics assumed for the field components. This choice is reasonable for modeling an unknown diffuser and analyzing its statistical properties. A more realistic description is either a Gaussian distribution with a constant added field to account for retroreflection components,[9] or a noncircular Gaussian distribution[2] as expected for a diffuser whose depth of roughness is not large compared to the wavelength.

In both cases, the second-order statistics is quite difficult to treat in exact form. However, for the case of constant added field it can be found that, besides the results reported[2] on the intensity, the first-order conditional probability density $p(\vartheta|I)$ and variance $\sigma_\vartheta^2|_I$ of the phase are given by expressions formally identical with Eqs. (15) and (16), where $z$ is change in $\sqrt{I_0 I}/2\sigma^2$ and $\vartheta_1 - \vartheta_2$ in $\vartheta$, and $I_0$ is the intensity of the constant added field. Moreover, for high coherence ($\mu \approx 1$) and large added field ($I_0 \gg I_1, I_2$), one can expand the joint probability and obtain the conditional phase variance as $\sigma_\vartheta^2|_{I_1,I_2} = \zeta^2/(I_0/\sigma^2)$, a result resembling Eq. (17), which is useful as a boundary estimate of phase error and NED.


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