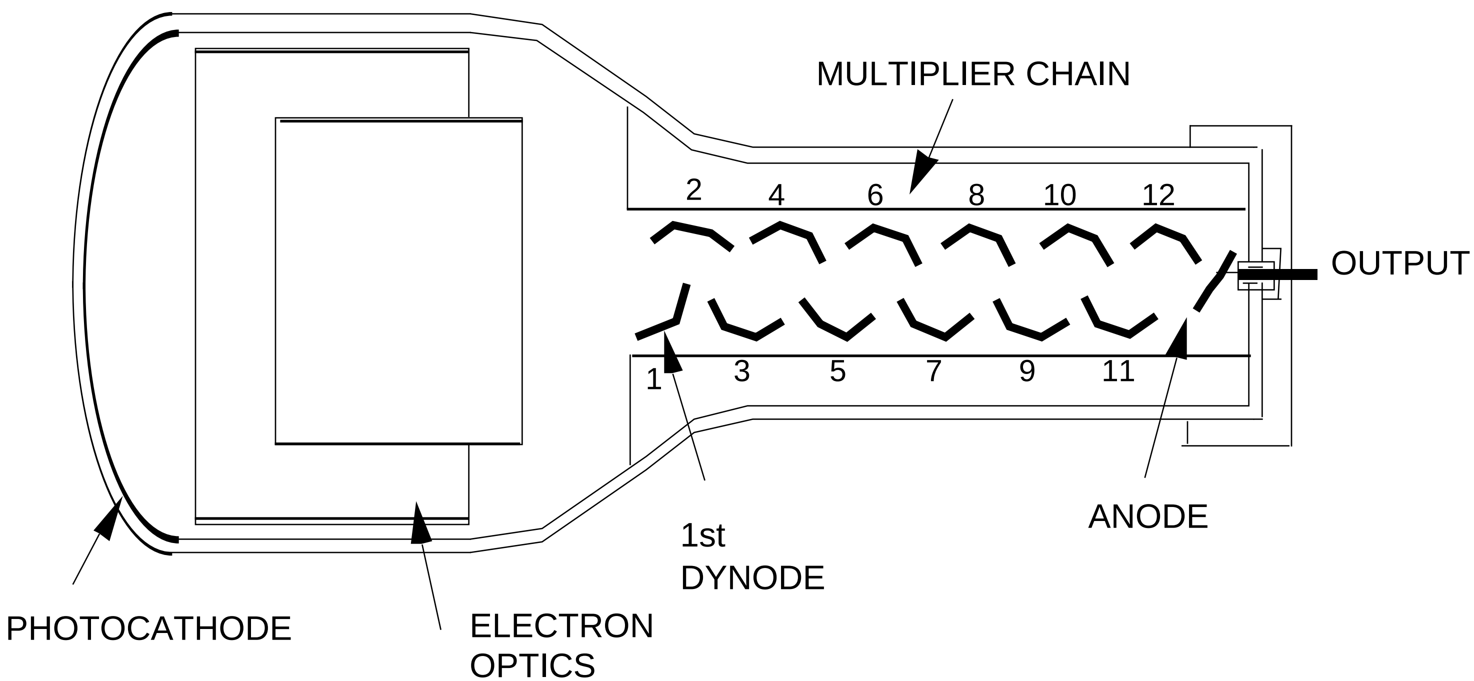
Photomultiplier (PMT) basic elements



from:"Photodetectors", by S.Donati, Prentice Hall 2000

PMT formats

PMT formats range from 1/2”-dia. tubes to 60 cm and more, window shape may be circular, hemispherical or hexagonal number of dynodes is from 6 to 12 (max 14) with any S-type photocathode to cover the 150 to 1200 nm spectral range

from:”Photodetectors”, by S.Donati, Prentice Hall 2000

Typical values

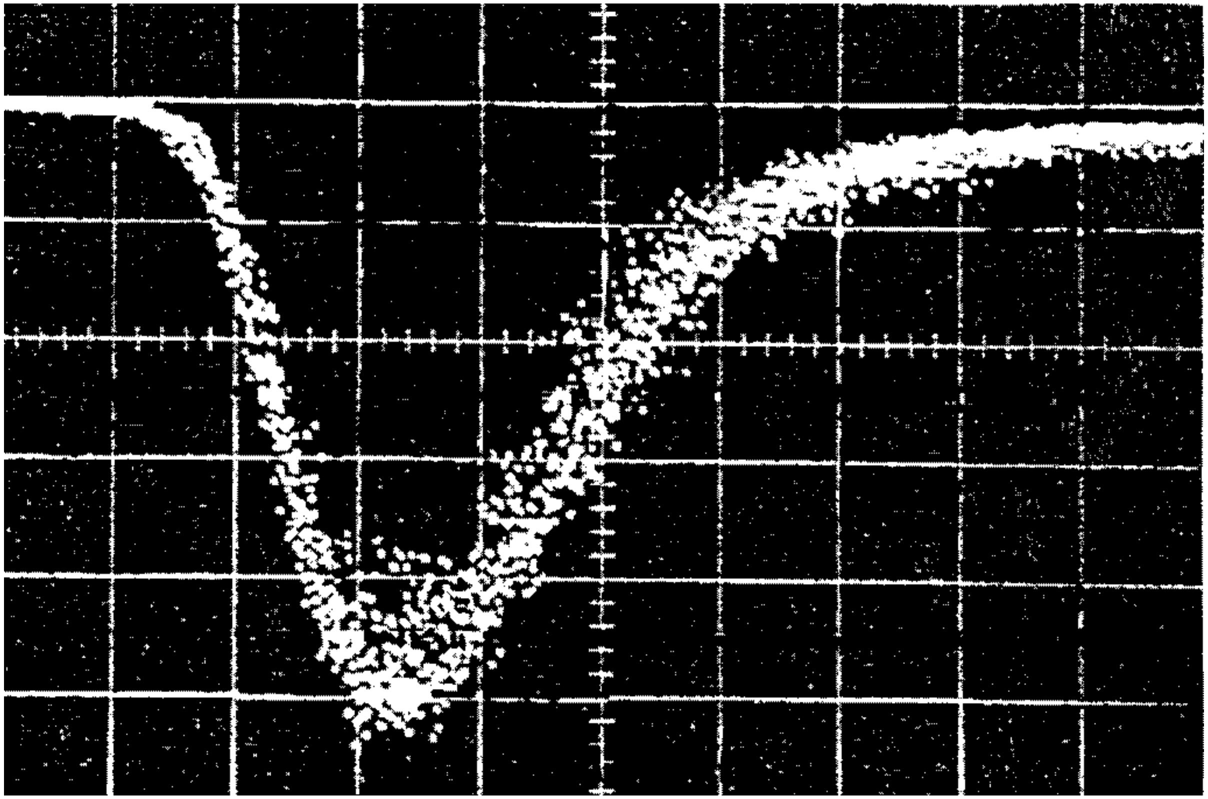
Typically, at 200 V of inter-dynode voltage, the dynode gain is g=4 per stage, the transit time is 2.5 ns per stage and its time-spread is 0.5 ns

For example, with 12-stage PMTwe have:

- a *gain*, $G = 4^{12}$ = **$1.6\ 10^{7}$**
- an (unessential) pulse *delay*, 2.5x12 = **30 ns**
- a pulse *duration* due to spread, $\tau = 0.5 \times \sqrt{12}$= **1.8 ns** (the impulse response)
- a *current peak*, $I = eG/\tau = 1.6\ 10^{-19}\ 1.6\ 10^{7} / 1.8\ 10^{-9}$ = **1.4 mA** (well detectable)

from:"Photodetectors", by S.Donati, Prentice Hall 2000

Single Electron Response (SER)



vert scale: 0.5 mA/div; hor. scale: 1 ns/div

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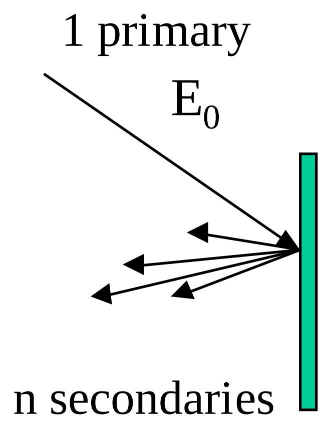
Secondary multiplication



For one primary electron impinging on a dynode with energy E_0 we find n secondary electrons emitted.

Number is not fixed, varies from primary to primary.

Probability of having n secondaries is Poisson distributed,
$p(n) = g^n e^{-g} / n!$
where g is the mean value (or mean gain of the dynode)

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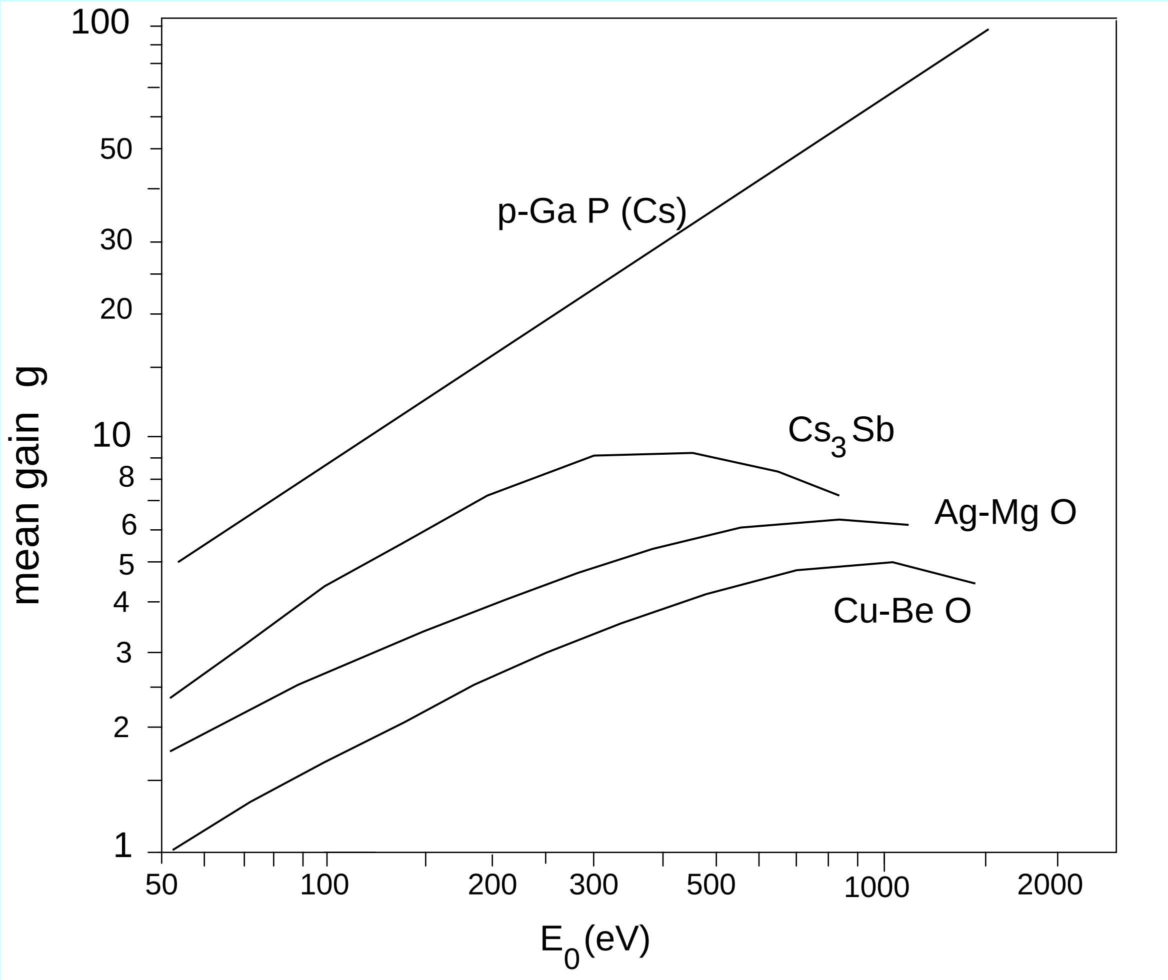
Dynode Materials



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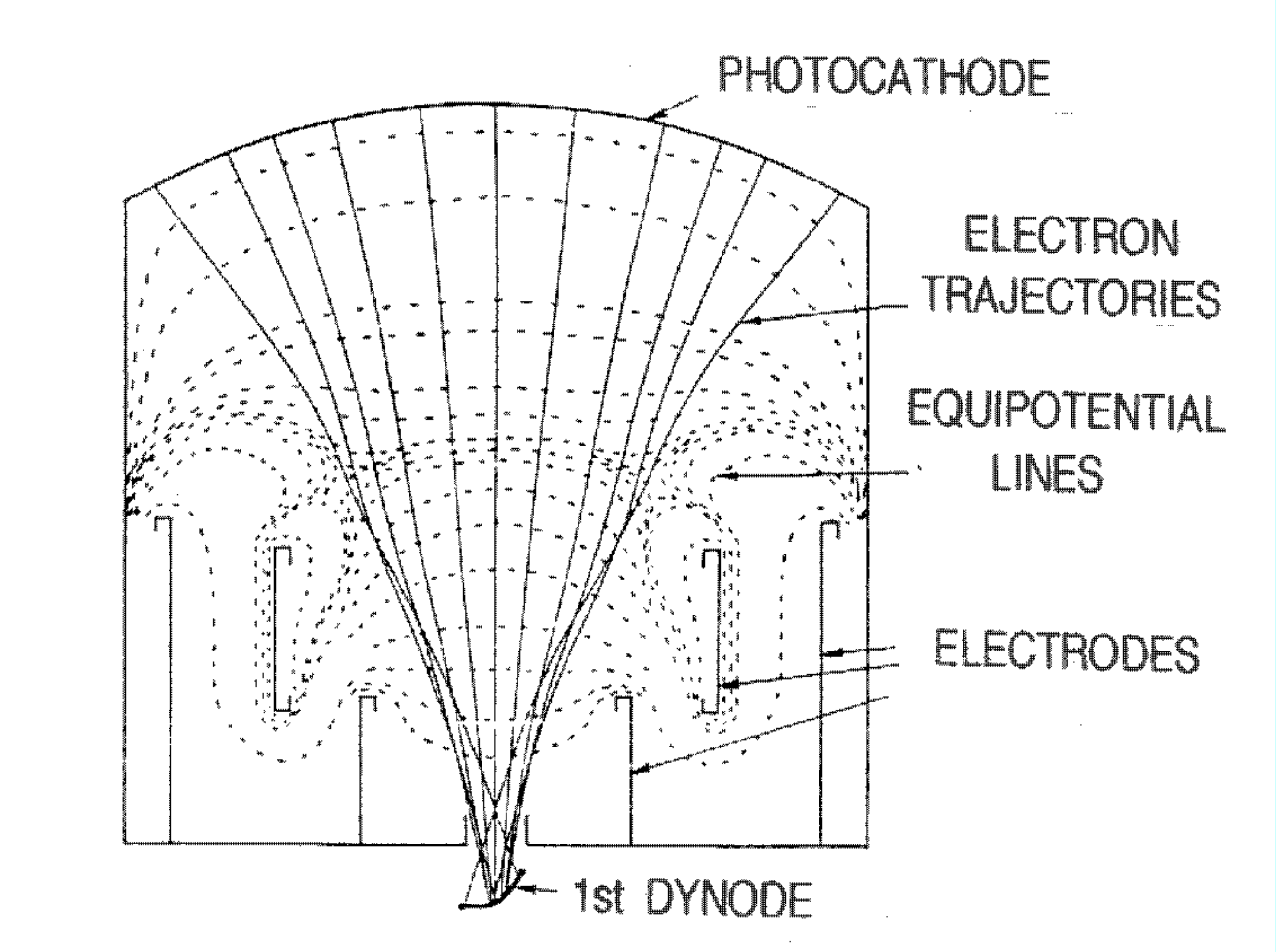
The electron optics



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Excursus: Design of the electron optics

Given a tentative structure and electrode voltages, one looks for the potential distribution $V(r,z)$:

$$\partial^2 V(r,z)/\partial z^2 + (1/r)\, \partial V(r,z)/\partial r + \partial^2 V(r,z)/\partial r^2 = 0$$

(boundary conditions: $V(r,z)=V_{ext}$, the applied voltages).

As $n(r,z) = \sqrt{V(r,z)}$ is the electrostatic index of refraction, the ray equation giving the trajectory $r=r(z)$ can then be integrated:

$$2V(r,z)\, \partial^2 r(z)/\partial z^2 + \partial V(r,z)/\partial z\, \partial r(z)/\partial z + (r/2)\partial^2 V(r,z)/\partial z^2 = 0$$

(initial conditions: $r=r_0$ (position) and $dr/dz=\tan\phi_0$ (slope) at $z=0$).

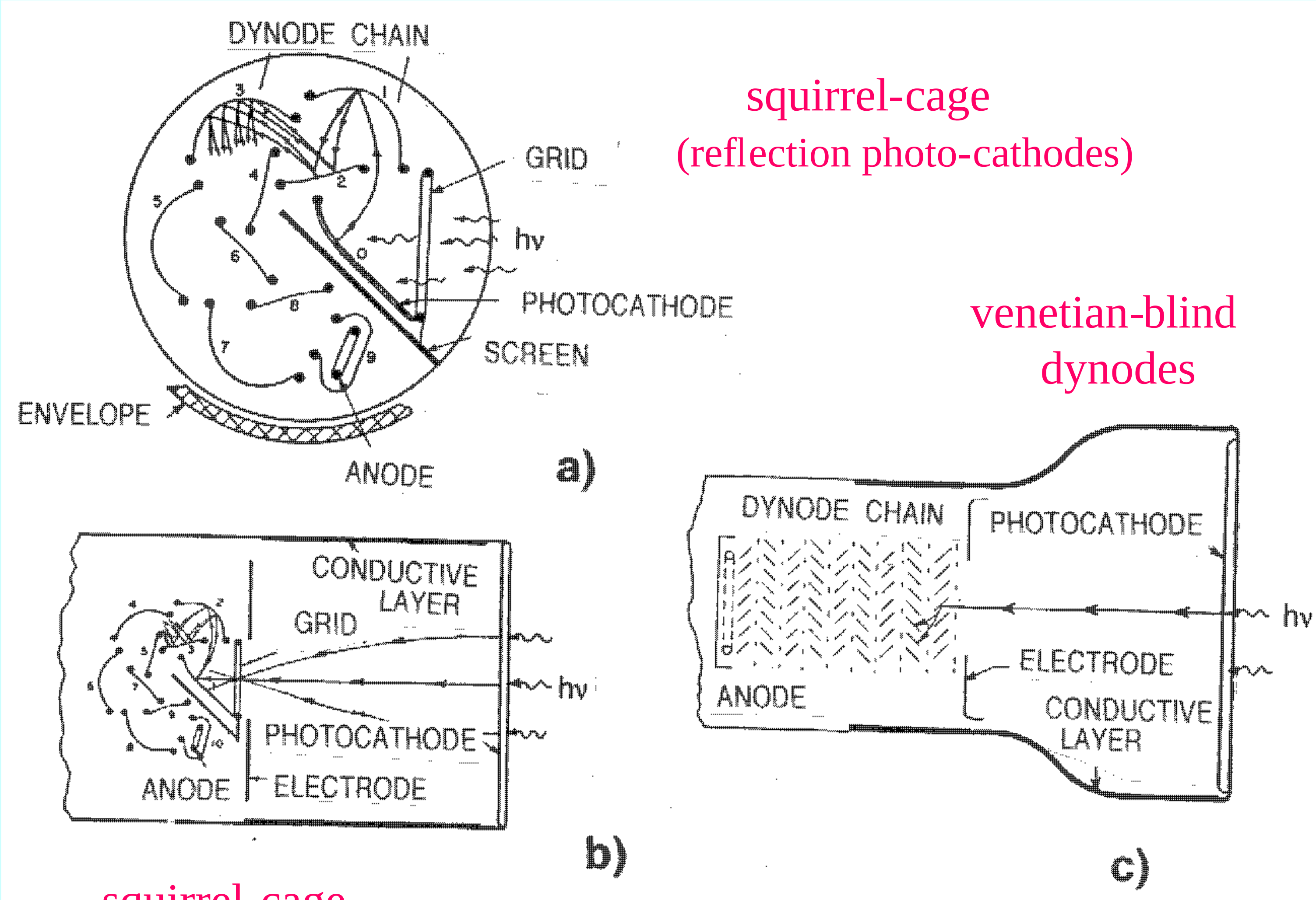
Solution yields the output coordinate $r=r(L)$ of the ray.
Repeating ray calculations with different $\tan\phi_0$, we get the confusion distribution

$$r=r(L,r_0)$$

of the electron-optics for entrance at $[z=0, r=r_0]$ in the object plane.
Many steps of optimization are repeated on the full image, with slight changes the electrodes shapes and voltages so as to minimize aberrations all over the image-field.

from:"Photodetectors", by S.Donati, Prentice Hall 2000

Common PMT structures



squirrel-cage
(reflection photo-cathodes)

venetian-blind
dynodes

squirrel-cage
(transmission photocathodes)

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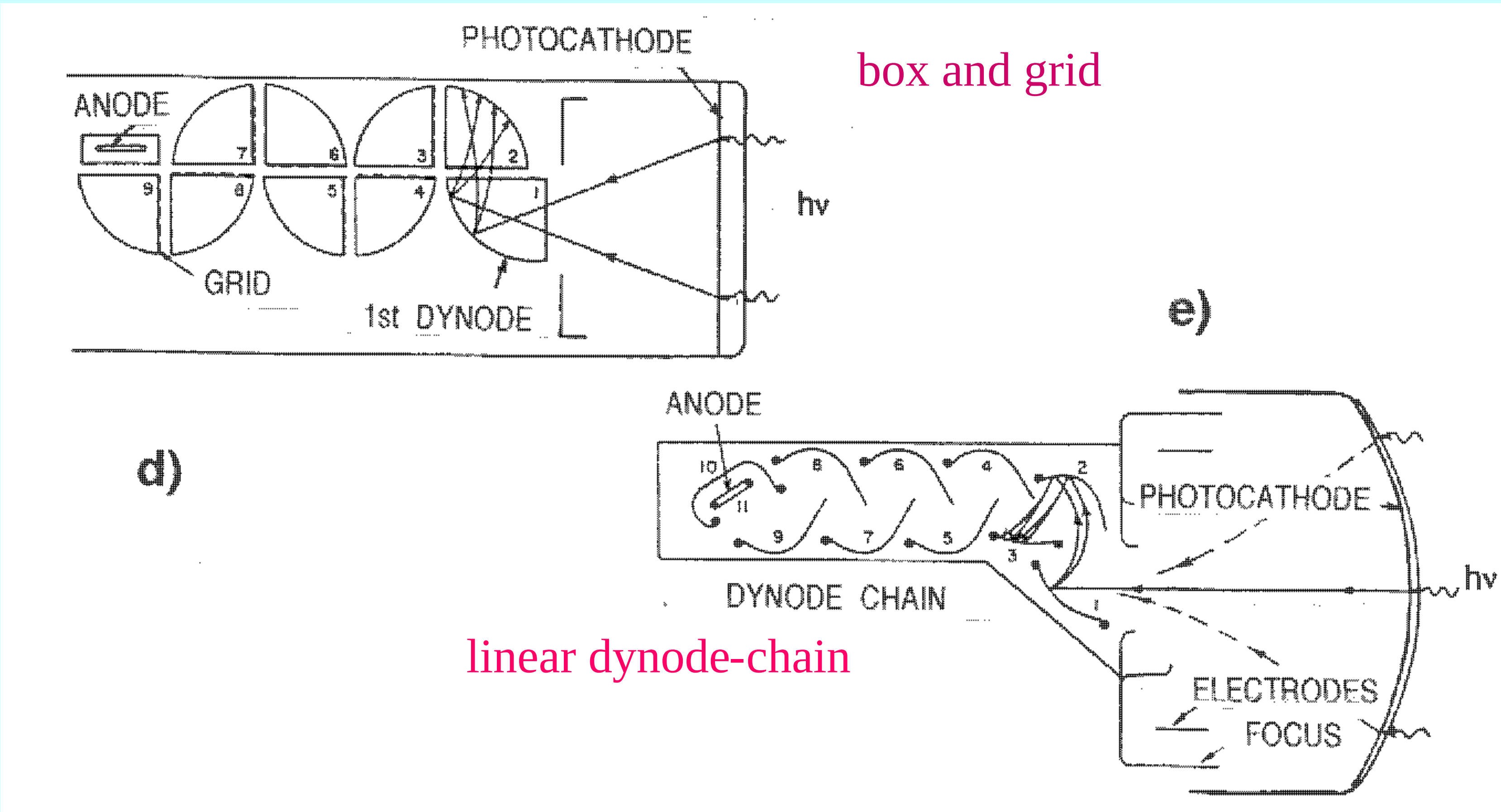
Common PMT structures (cont'd)



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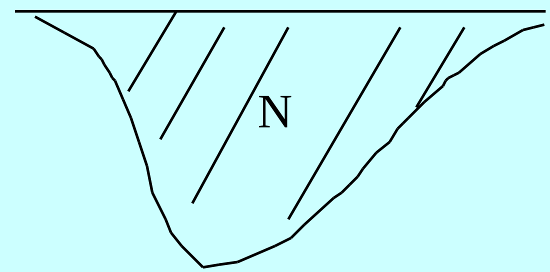
Types of PMT responses to be considered

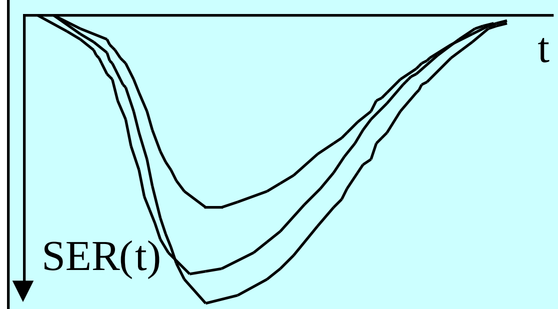
•***Charge* (or integral) response**:is the number N of electrons collected at the anode for one photoelectron impinging on the first dynode (*multiplier chain response*) or one photon detected by the photocathode (*PMT response*). Extension: to a light pulse containing R photons (*scintillation counter response*). For all statistics, the random variable N is characterized by mean $\langle N \rangle$ and variance $\sigma_N^2 = \langle [N - \langle N \rangle]^2 \rangle$.

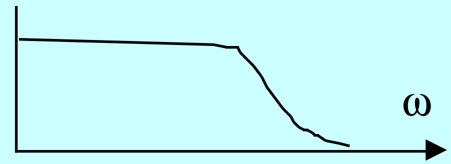
•***Impulse* (or current) response** in the time domain: is the time-dependent output SER (*single electron response*) for an electron at t=0 on the first dynode. We will calculate: the mean $\langle SER(t) \rangle$ and variance $\sigma_{SER}^2(t)$, the output current i(t) for 1 or R photon/s detected at t=0, and the response to an optical signal I(t) carrying R photons. A variant is the ***correlation response***.

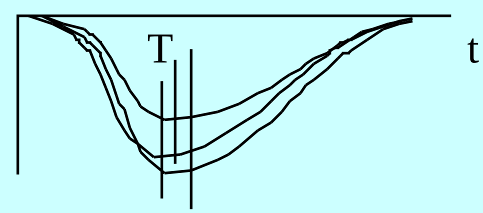
•**Response in the *frequency domain***: is the transfer function F(ω) of the PMT, given by the mean and noise spectral density s(ω) for different inputs.

•***Time localization***: is about time measurements, i.e., the delay T between a short light pulse detected at t=0 and the output anode pulse, where $\langle T \rangle$ is the mean delay and the variance σ_T^2 yields the accuracy of time localization supplied by the PMT.









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Charge response

multiplication at the dynode (fixed input)

for 1 impinging electron

$\langle m \rangle = \langle n \rangle = g,$

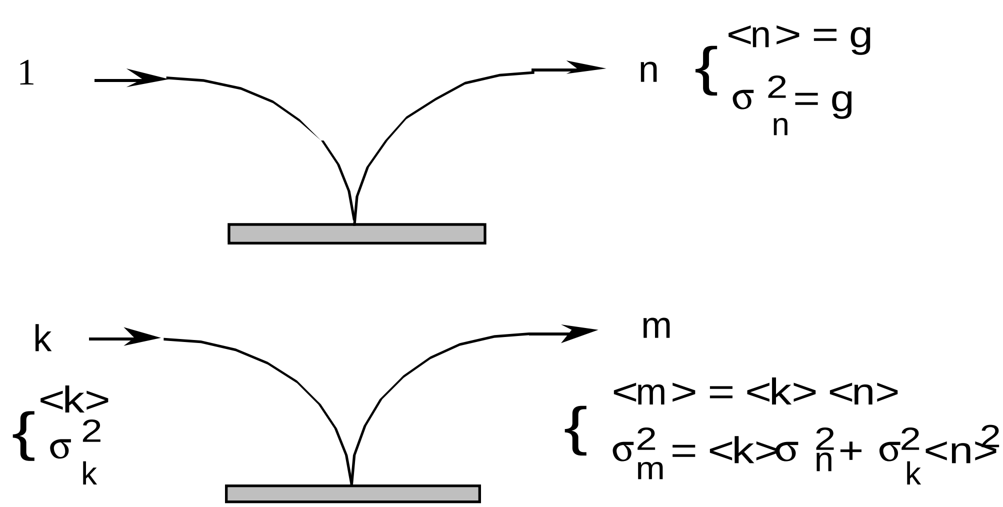
$\sigma_n^2 = g$

for K impinging electrons

$\langle m \rangle = K \langle n \rangle = Kg,$

$\sigma_m^2 = K\sigma_n^2 = Kg$

model of dynode multiplication (random input)



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Charge response (cont'd)

Applying iteratively these equations we get:

dynode	mean $\langle N \rangle$	variance σ_N^2
1°	g_1	g_1
2°	$g_1 g_2$	$g_1 g_2 + g_1 g_2^2$
3°	$g_1 g_2 g_3$	$g_1 g_2 g_3 + g_1 g_2 g_3^2 + g_1 g_2^2 g_3^2$
4°	$g_1 g_2 g_3 g_4$	$g_1 g_2 g_3 g_4 + g_1 g_2 g_3 g_4^2 + g_1 g_2 g_3^2 g_4^2 + g_1 g_2^2 g_3^2 g_4^2$
. .	. . . . .	. . . .
n-th	$g_1 g_2 \cdots g_n$	$\sum_{i=1,n} g_1 g_2 \cdots g_i \; [g_{i+1} \cdots g_n]^2$

Whence : $\langle N \rangle = g_1 g_2 \cdots g_n$

$$\sigma_N^2 = \langle N \rangle^2 \sum_{i=1,n} 1 / (g_1 g_2 \cdots g_i)$$

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Charge response (cont'd)

Letting all dynode gains g equal except the first g_1 we have:

$$\sigma_N^2 = \langle N \rangle^2 (1/g_1)\,[1+1/g+1/g^2+ \cdots +1/g^{n-1}]$$

$$\approx \langle N \rangle^2 /[g_1(1 - 1/g)] = \langle N \rangle^2 \varepsilon_A^2$$

where

$$\varepsilon_A^2 = g /[g_1(g -1)]$$

is called the *multiplier variance*, because it just gives the relative variance of the charge,

$$\sigma_N^2 / \langle N \rangle^2 = \varepsilon_A^2$$

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Charge response (cont'd)

For *exactly* R photoelectrons arriving to the 1st dynode,

$$\langle N_R \rangle = g_1 g_2 \cdots g_n R = \langle N \rangle R$$

$$\sigma_{NR}^2 = \sigma_N^2 R = \langle N_R \rangle^2 \varepsilon_A^2 R$$

and the relative variance is:

$$\sigma_{NR}^2 / \langle N_R \rangle^2 = \varepsilon_A^2 / R$$

i.e., it improves as 1/R.

Last, consider the R photoelectrons be Poisson-distributed, as due to F photons detected by the photocathode with quantum efficiency η, so that

$$R = \eta F$$

photoelectron are emitted.

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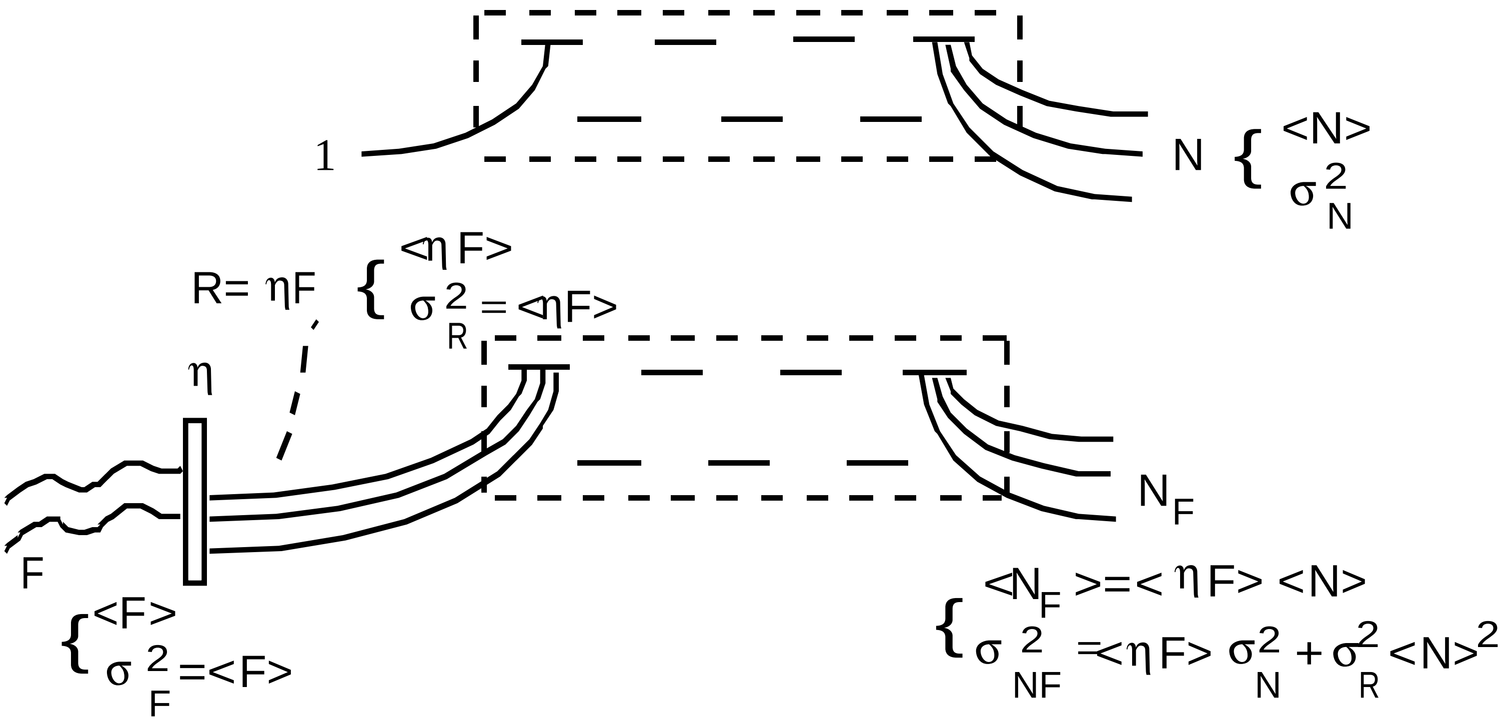
Charge response (cont'd)

If F is Poisson distributed, also R= ηF is Poisson distributed, whence $\sigma_R^2 = \langle R \rangle$. The statistical compounding rules are schematized below:



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Charge response (cont'd)

By applying the compounding rules, we get:

$$\langle N_F \rangle = \langle \eta F \rangle \langle N \rangle$$

$$\sigma_{NF}^2 = \langle \eta F \rangle \sigma_N^2 + \sigma_R^2 \langle N \rangle^2 = \langle N_F \rangle^2 (1+ \varepsilon_A^2) / \langle \eta F \rangle$$

As the relative variance carried by the packet of photons F is:

$$\sigma_F^2 / \langle F \rangle^2 = 1/\langle F \rangle$$

the photomultiplier adds a worsening by a factor:

$$NF^2 = [\sigma_{NF}^2 / \langle N_F \rangle^2] / [\sigma_F^2 / \langle F \rangle^2] = (1+\varepsilon_A^2)/\eta$$

NF^2 is called the *noise figure* and, is nicely low .! .!

[for example, ε_A^2=0.33 for g=4 and with η =0.33 we have NF^2=4]

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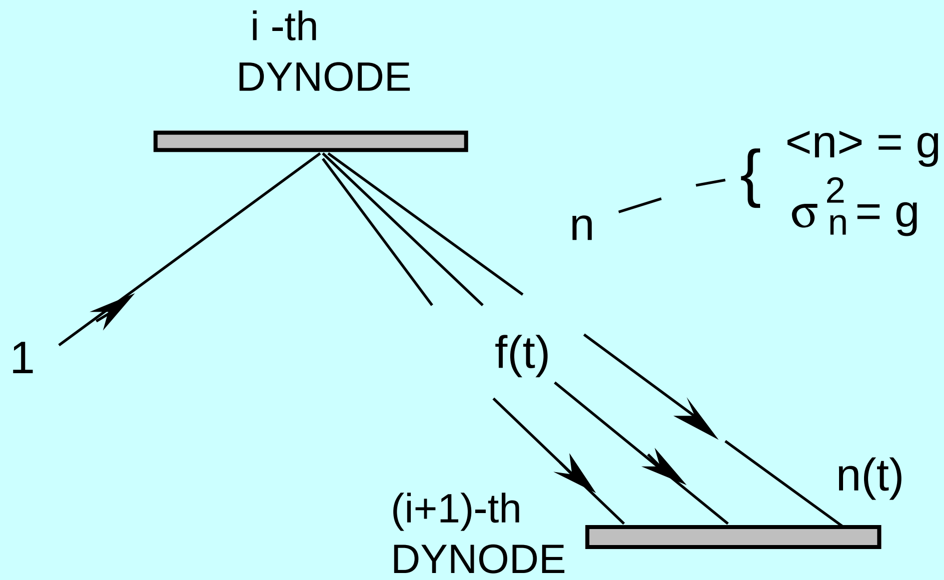
Current response



The process of multiplication is now time-dependent and inter-dynode flights are schematized by a function f(t), the pdf of having a time-of-flight t–t+dt when the electron leaves the i-th-dynode at time t=0.
The number of secondaries n_T again electrons follows the Poisson statistics. Each electrons from the i-th dynode covers the interdynode flight independently

Choices for f(t): Gaussian: $f(t) = [\sqrt{(2\pi)}\sigma_t]^{-1} \exp[-(t - t_0)^2/2\sigma_t^2]$

or polynomial-exponential: $f(t) = [\Gamma(m+1)\sigma_t^{\,m+1}]^{-1}\, t^m \exp(-t/\sigma_t)$

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Current response (cont'd)

A random current will be written as

$$I(t) = e\, N\, f(t)$$

where e is the electron charge,
N is the random total charge,
f(t) the time-dependence

(for simplicity, we will omit e from now on)

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Current response (cont'd)

mean current at the (i+1)-th dynode:

$$\langle n(t) \rangle = \langle n_T \rangle \, f(t)$$

where n_T is the (random) total number of secondaries [$n_T = \int_{0-\infty} \langle n(t) \rangle \, dt$], with mean and variance:

$$\langle n_T \rangle = g, \quad \sigma_{nT}^2 = g,$$

and as $f(t) = f_i(t)$ we have:

$$\langle n(t) \rangle = g \, f_i(t)$$

To compute the variance, we shall introduce the generatlized concept of covariance

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Current response (cont'd)

Covariance function $K_n(t,t')$:

$$K_n(t,t') = \langle [n(t)-\langle n(t)\rangle][n(t')-\langle n(t')\rangle]\rangle$$

for t=t' the covariance becomes the variance, $K_n(t,t)=\sigma_i^2(t)$; and, if K is dependent on $t-t'=\tau$ only, it becomes the correlation function $\rho(\tau)$.

For a process with a sequence of n_T independent electrons with arrival times f(t), one can find that the covariance is:

$$K_n(t,t') = [(\sigma_{nT}^2/\langle n_T\rangle) + \langle n_T\rangle - 1]\, f(t)\, \delta(t'-t)$$

where $\delta(t'-t)$ is the Dirac delta-function.

For a Poisson distributed n_T, we have:

$$K_n(t,t') = g\, f(t)\, \delta(t'-t)$$

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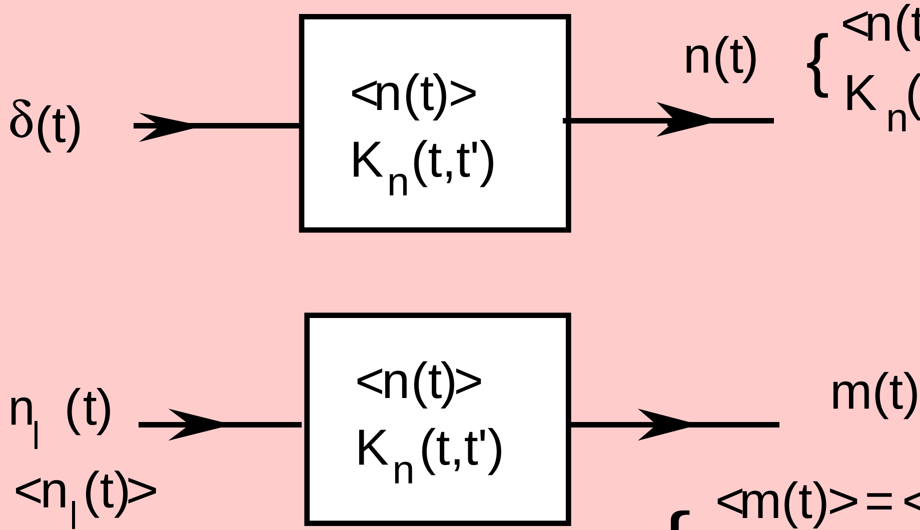
Current response (cont'd)



$$\begin{cases} \langle n(t)\rangle = g\, f(t) \\ K_n(t,t') = g\, f(t)\, \delta(t'-t) \end{cases}$$

$$\begin{cases} \langle n_l(t)\rangle \\ K_l(t,t') \end{cases}$$

$$\begin{cases} \langle m(t)\rangle = \langle n_l(t)\rangle * \langle n(t)\rangle \\ K_m(t,t') = \langle n_l(t)\rangle \delta(t'-t) ** K_n(t,t') + \\ \qquad + K_l(t,t') ** \langle n(t)\rangle \langle n(t')\rangle \end{cases}$$

Schematization of a random time-dependent multiplication through mean and covariance (top) and rule of composition of signal and delta response

from:"Photodetectors", by S.Donati, Prentice Hall 2000

Current response (cont'd)

Applying iteratively the composition rules we have :

dynode	mean $\langle i(t) \rangle$	covariance $K_i(t,t')$
1st	$g_1 f_1(t)$	$g_1 f_1(t)\delta(t'-t)$
2nd	$g_1 g_2 f_1(t) * f_2(t)$	$g_1 f_1(t)\delta(t'-t) ** g_2 f_2(t)\delta(t'-t) + g_1 f_1(t)\delta(t'-t) ** g_2^2 f_2(t) f_2(t')$
		$= g_1 g_2 f_1(t) * f_2(t)\delta(t'-t) + g_1 g_2^2 f_1(t) * [f_2(t) f_2(t')]$
3rd	$g_1 g_2 g_3 f_1(t) * f_2(t) * f_3(t)$	$g_1 g_2 g_3 f_1(t) * f_2(t) * f_3(t)\delta(t'-t) +$
		$+ g_1 g_2 g_3^2 f_1(t) * f_2(t) * [f_3(t) f_3(t')] +$
		$+ g_1 g_2^2 g_3^2 f_1(t) * [f_2(t) * f_3(t)][f_2(t') * f_3(t')]$
. .	. . . . .	. . . .
n-th	$g_1 g_2 \cdots g_n\, f_1(t) * f_2(t) * \cdots * f_n(t)$	
		$\sum g_1\, g_2 \cdots g_i\, [g_{i+1} \cdots g_n]^2\, f_1(t) * f_2(t) * \cdots * f_i(t) *$
		$[f_{i+1}(t) * \cdots * f_i n(t)][f_{i+1}(t') * \cdots * f_n(t')]$

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Current response (cont'd)

This response is the SER (*single-electron-response*).

Mean value is:

$$\langle SER(t) \rangle = g_1 g_2 \cdots g_n f_1(t) * f_2(t) * \cdots * f_n(t)$$

Also, as $\langle N \rangle = g_1 g_2 \cdots g_n$:

$$\langle SER(t) \rangle = \langle N \rangle \; f_1(t) * f_2(t) * \cdots * f_n(t)$$

The covariance is:

$$K_{SER}(t,t') = \sum_{i=1,n} g_1 g_2 \cdots g_i [g_{i+1} \cdots g_n]^2 f_1(t) * f_2(t) * \cdots * f_i(t) * [f_{i+1}(t) * \cdots * f_n(t)][f_{i+1}(t') * \cdots * f_n(t')]$$

and the SER variance (for t=t') is :

$$\sigma_{SER}^2(t) = \sum_{i=1,n} g_1 \cdots g_i [g_{i+1} \cdots g_n]^2 f_1(t) * f_2(t) * \cdots * f_i(t) * *[f_{i+1}(t) * \cdots * f_n(t)]^2$$

Current response (cont'd)

Rigid SER approximation:

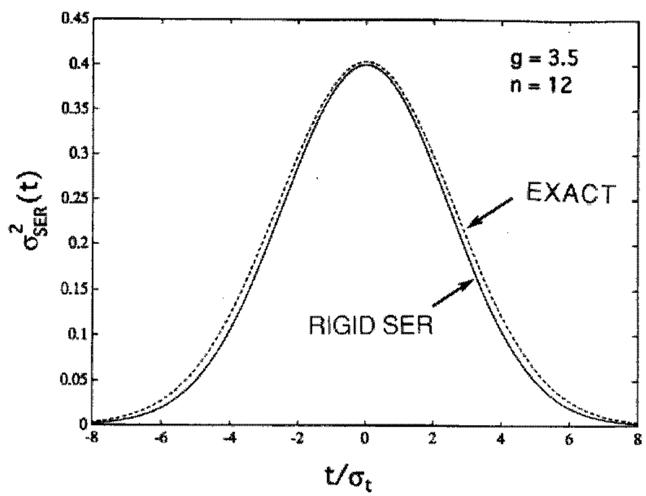
$$f_1(t)_* \cdots {}_* f_i(t)_* [f_{i+1}(t)_* \cdots {}_* f_n(t)]^2 \approx [f_1(t)_* f_2(t)_* \cdots {}_* f_n(t)]^2$$

then we have:

$$K_{SER}(t,t') \approx \sum 1/(g_1 g_2 \cdots g_i)\, \langle SER(t) \rangle \langle SER(t') \rangle$$

$$= \varepsilon_A{}^2 \langle SER(t) \rangle \langle SER(t') \rangle$$

$$\sigma_{SER}{}^2(t) \approx \varepsilon_A{}^2 \langle SER(t) \rangle^2$$



An example of the accuracy of the rigid SER approximation for n=12 stages and a Gaussian time-of-fight distribution

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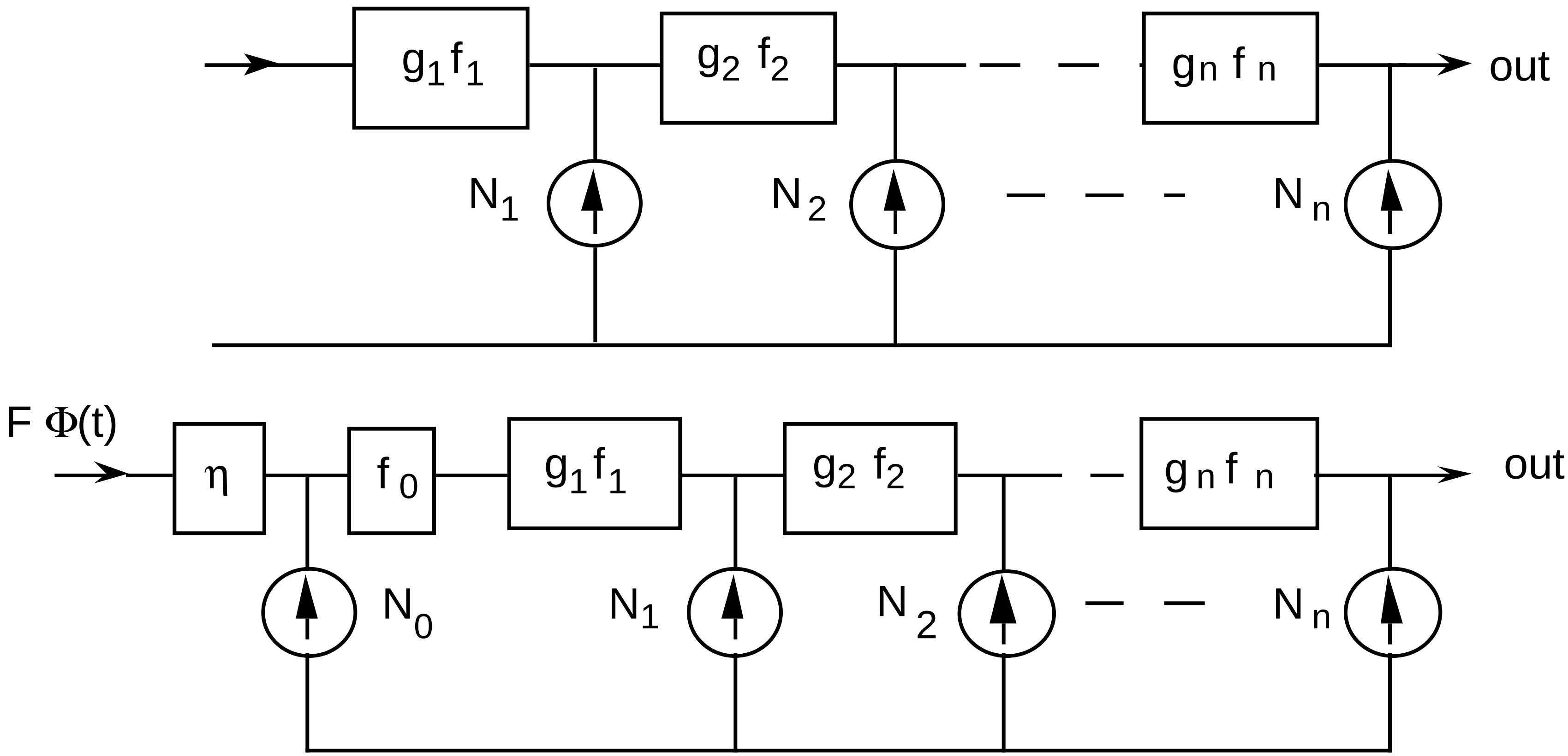
Current response (cont'd)



Noise equivalent circuit of the dynode multiplying chain (top) and of the photomultiplier (bottom). Noisegenerators N_i sum to the mean signal a white noise of intensity equal to the mean signal

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Current response (cont'd)

Case of a fixed number R of photoelectrons in a single short pulse with $\langle L(t)\rangle = \langle F\rangle \Phi(t)$:

$$\langle I(t)\rangle = R\, f_0(t) * \langle SER(t)\rangle$$

$$K_I(t,t') = R\, f_0(t)\delta(t'-t)_* * K_{SER}(t,t') \approx R\, f_0(t) * \langle SER(t)\rangle \langle SER(t')\rangle\, \varepsilon_A^2$$

$$\sigma_I^2(t) \approx R\, f_0(t) * \langle SER(t)\rangle^2 \varepsilon_A^2$$

Also in this case, the relative variance of the response, given by:

$$\sigma_I^2(t)/\langle I(t)\rangle^2 = [\, f_0(t) * \langle SER(t)\rangle^2 \varepsilon_A^2] \,/\, R\, [f_0(t) * \langle SER(t)\rangle\,]^2$$

can be approximated, on the rigid SER approx., to:

$$\sigma_I^2(t) / \langle I(t)\rangle^2 \approx \varepsilon_A^2/R$$

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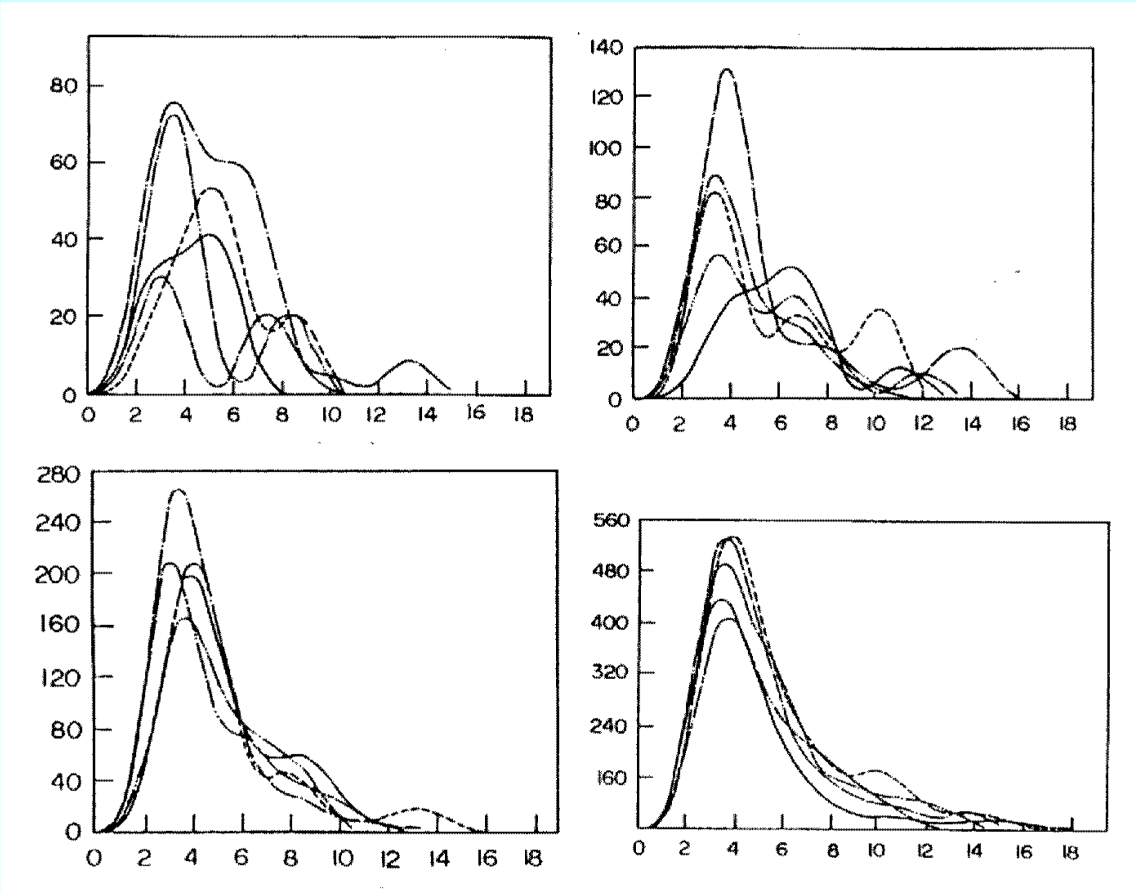
Current response (cont'd)



Some samples of the anode response of the photomultiplier for R= 10, 20 (top), 50, 100 (bottom) photoelectrons emitted at t=0, calculated by a Montecarlo method. Horizontal scale: time in ns; vertical scale: current in relative units

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Correlation response

The correlation function $\rho(\tau)$ of a light field $L(t)$ is defined as:

$$\rho_L(\tau) = \int_{t=0-\infty} L(t)\, L(t+\tau) dt$$

If $L(t)$ is a train of photons occurring at random times t_k, i.e.:

$$L(t) = h\nu \sum_{k=-\infty,\, +\infty} \delta(t-t_k)$$

developing ρ_L the correlation is found as:

$$\rho_L(\tau) = (h\nu)^2\{F\, \delta(\tau) + F^2[1+\gamma(\tau)]\}$$

where $\gamma(\tau)$ is the reduced correlation function,

$$\gamma(\tau) = \langle\, [L(t)-\langle\, L(t)\rangle\,]\, [L(t+\tau)-\langle\, L(t+\tau)\rangle\,]\rangle\ /\ \langle\, L(t)\rangle^{\,2}$$

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Correlation response (cont'd)

The output current correlation $\rho_I(\tau)$ of I(t), can then be computed and the result is :

$$\rho_I(\tau) = (\eta e)^2 F \rho_{\Delta SPR}(\tau) + (\eta e)^2 F^2 [1+\gamma(\tau)] * \rho_{\langle SPR \rangle}(\tau)$$

where

$$\rho_{\langle SPR \rangle}(\tau) = \int_{t=0-\infty} \langle SPR(t) \rangle \langle SPR(t+\tau) \rangle \, dt$$

is the autocorrelation associated with the mean waveform $\langle SPR(t) \rangle = f_0(t) * \langle SER(t) \rangle$ of the single photon response, and

$$\rho_{\Delta SPR}(\tau) = \rho_{\langle SPR \rangle}(\tau) + \int_{t=0-\infty} K_{SPR}(t,t+\tau) \, dt$$

is the autocorrelation of the SPR fluctuations, contained in the covariance K given by :

$$K_{SPR}(t,t+\tau) = \varepsilon_A^2 f_0(t) * \langle SER(t) \rangle \langle SER(t') \rangle$$

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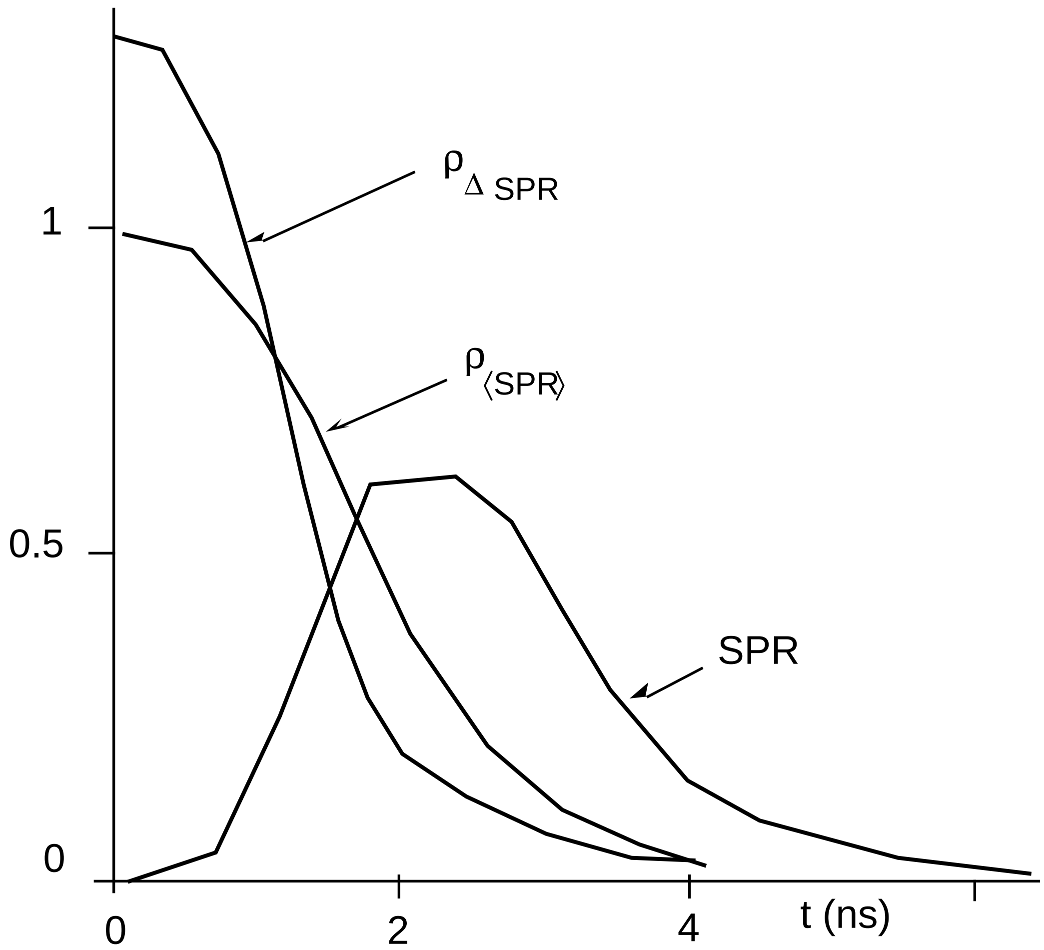
Correlation response (cont'd)



To describe the PMT correlation response, two functions are required, ρ(ΔSPR) and ρ⟨ SPR⟩

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Correlation response (cont'd)

When the correlation times are much longer, or $f_0(t)$ is short compared to the SER, a rigid SPR approximation is adequate and the previous equations become:

$$\rho_{\Delta SPR}(\tau) = (1+\varepsilon_A^2)\, \rho_{\langle SPR\rangle}(\tau)$$

$$\rho_I(\tau) = (\eta e)^2 (1+\varepsilon_A^2)\, F\rho_{\langle SPR\rangle}(\tau)+(\eta e)^2 F^2 [1+\gamma(\tau)]_* \rho_{\langle SPR\rangle}(\tau)$$

By deconvolution of $\rho_I(\tau)$ with $\rho_{\langle SPR\rangle}$, light correlations $\gamma(\tau)$ waveforms down to about 10% of SER duration can be recovered

from:"Photodetectors", by S.Donati, Prentice Hall 2000

Frequency Domain response

The transfer function $F(\omega)$ is the Fourier transform of the δ–response $U_\delta(t)$, and the noise power spectrum $S^2(\omega)= di^2/d\omega$ is the Fourier transform of $K_U(\tau)$:

$$\textit{frequency response}: \quad F(\omega) = \mathcal{F}\{\langle SER(t)\rangle\},$$

for a Gaussian waveform : $F(\omega)=\langle N\rangle \exp(-\omega^2 n\sigma_t^2/2+i\omega n t_0)$, whence a high frequency cutoff (at -3dB) of $f_T=0.83/2\pi\sigma_t\sqrt{n}= 0.13/\sigma_t\sqrt{n}$.

$$\textit{noise power spectral density}: S^2(\omega) = 2e\,(I_{ph}+I_d)\,(1+\varepsilon_A^2)\,|F(\omega)|^2$$

In the passband, the PMT is close to the quantum limit for $I_{ph0}\geq I_d$, i.e. for an input power larger than $P_{i0}\geq(h\nu/\eta e)I_d$. In this case, the S/N is:

$$(S/N)^2 = I_{ph} / 2eB(1+\varepsilon_A^2) = P/2(h\nu/\eta)B(1+\varepsilon_A^2)$$

$$\textit{noise figure}: NF^2=(1+\varepsilon_A^2)/\eta.$$

P_i is very small, even for small η, because of the very minute dark currents I_d of the photocathode, for ex., for I_d =1 fA and η=0.1, quantum-limited performance is at $P\geq P_{i0}$= 20 fW and is over the full frequency band.

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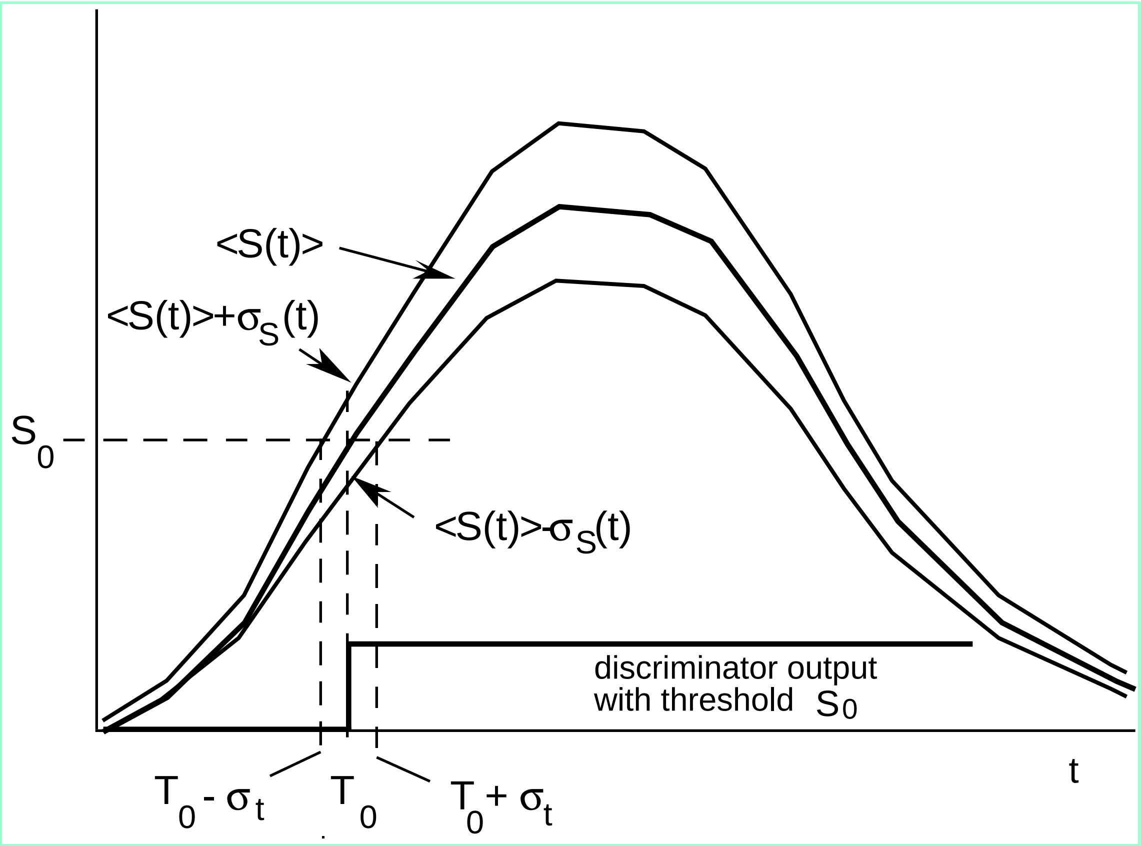
Time Sorting and Measurements



Time sorting can be simply performed by a threshold crossing at a suitable level S_o and taking T_o as the timing information carried by the signal

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Time Sorting and Measurements (cont'd)

Time variance for crossing of a fixed amplitude threshold:

$$\sigma_T^2(T_o) = \sigma_S^2(T_o)/\langle \,|dS/dt|_{t=To}\rangle^2$$

for the time measurement on SER:

$$\sigma_{T1}^2 = \varepsilon_A^2 \langle SER(t)\rangle^2 /[d/dt \langle SER(t)\rangle]^2$$

and for the time measurement on a fast pulse:

For the time measurement on SER:

$$\sigma_{T\delta}^2 = [(1+\varepsilon_A^2)/R]\{f_o(t) * \langle SER(t)\rangle \}^2/\{d/dt\, [f_o(t)*\langle SER(t)\rangle]\}^2$$

Example: for Gaussian-distributed time-of-flights $f_o, f_1, .., f_n$, each with a variance σ_t , using a threshold at the maximum slope of the SER we have:

$$\sigma_{T\delta}^2 = [(1+\varepsilon_A^2)/R](n+1)\sigma_t^2$$

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Time Sorting and Measurements (cont'd)

The variance of the SER centroid, as the weighted sum of interdynode-flight uncertainties is:

$$\sigma_{Tc}^2 = \sigma_{t0}^2 + (1/g_1)\sigma_{t1}^2 + (1/g_1g_2)\sigma_{t2}^2 + \ldots + (1/g_1g_2\ldots g_n)\sigma_{tn}^2$$

and for equal $\sigma_{ti} = \sigma_t$:

$$\sigma_T^2 \approx \sigma_{t0}^2 + \varepsilon_A^2\,\sigma_t^2$$

and in the case of $R = \langle \eta F \rangle$ photoelectrons:

$$\sigma_{Tc}^2 = [\sigma_{t0}^2 + (1/g_1)\sigma_{t1}^2 + (1/g_1g_2)\sigma_{t2}^2 + \ldots + (1/g_1g_2\ldots g_n)\sigma_{tn}^2]/R$$

$$= [\sigma_{t0}^2 + \varepsilon_A^2\,\sigma_t^2]/R$$

Written as $\sigma_{Tc}^2 = [1+\varepsilon_A^2]\sigma_t^2/R$ the centroid variance is smaller by a factor (n+1) as compared to the fixed-threshold crossing.

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