Chapter 5

TIME-VARYING ELECTROMAGNETIC FIELD
5.1 Maxwell's equations in differential form

In general, the presence of a movable charge distributed with density \( \rho \) (Cm\(^{-3}\)) and of an impressed current density \( \vec{J}_0 \) (Am\(^{-2}\)) variable with time gives origin to the electromagnetic field described by the following time-dependent vectors:

- \( \vec{D} \) electric displacement (Cm\(^{-2}\))
- \( \vec{E} \) electric field intensity (Vm\(^{-1}\))
- \( \vec{B} \) magnetic induction (T)
- \( \vec{H} \) magnetic field intensity (Am\(^{-1}\))
- \( \vec{J} \) current density (Am\(^{-2}\))

As far as the origin of current density is concerned, the following remark can be put forward. In a solid or liquid medium the conduction current density is a function of \( \vec{E} \)

\[
\vec{J} = \vec{J}(\vec{E})
\]  
(5.1.1)

For a linear medium the above function becomes

\[
\vec{J} = \sigma \vec{E}
\]  
(5.1.2)

Another kind of current is originated by the movement of free ions and electrons (e.g. in gases or vacuum). This convection current density is expressed by the formula

\[
\vec{J} = \rho_+ u_+ + \rho_- u_-
\]  
(5.1.3)

where \( \rho_+ \) and \( \rho_- \) are positive and negative charge densities, respectively, while \( u_+ \) and \( u_- \) are the relevant velocities of positive and negative free charges.

Finally, the displacement current density is defined as

\[
\vec{J} = \frac{\partial \vec{D}}{\partial t}
\]  
(5.1.4)

Considering the principle of charge conservation in any point of the domain, the following equation always holds (charge continuity equation)

\[
\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0
\]  
(5.1.5)
The coupled electric and magnetic fields influence a charge \( q \) (C) by exerting a mechanical force \( \vec{F} \) (N) on it (Lorentz's equation)

\[
\vec{F} = q (\vec{E} + \vec{u} \times \vec{B})
\]

(5.1.6)

where \( \vec{u} \) is the velocity of the charge with respect to the magnetic field. In particular, the term \( q\vec{E} \) modifies the value of velocity, while the term \( q\vec{u} \times \vec{B} \) modifies also the direction of velocity.

In a simply-connected domain \( \Omega \) with boundary \( \Gamma \) filled in by a linear medium characterized by permittivity \( \varepsilon \), permeability \( \mu \) and conductivity \( \sigma \), the time-varying electromagnetic field is described by the following equations:

**Faraday's equation (5.1.7)**

\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}
\]

Gauss's electric equation (5.1.8)

\[
\nabla \cdot \vec{D} = \rho
\]

Ampère's equation (5.1.9)

\[
\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}
\]

Gauss's magnetic equation (5.1.10)

In a three-dimensional domain, the above equations represent a set of eight scalar equations to which constitutive relations (2.2.1), (2.3.1), (2.4.1) must be added.

In total, fifteen scalar unknowns (i.e. field components) have to be determined, subject to suitable boundary conditions.

The system of eight plus nine equations can be solved since there are two relations among the unknowns which are automatically satisfied. One is (5.1.5) and the other comes from (5.1.7) and (5.1.8). In fact, taking the divergence of (5.1.9) and the time derivative of (5.1.8), continuity equation (5.1.5) follows. Similarly, taking the divergence of (5.1.7) and the time derivative of (5.1.10), one obtains an identity.

It should be remarked that in (5.1.9), in general,

\[
\vec{J} = \vec{J}_0 + \sigma \vec{E} + \mu \sigma \vec{u} \times \vec{H}
\]

(5.1.11)

where \( \vec{J}_0 \) is the term impressed by an external source, while the last term of the right-hand side takes into account the current density due to motional effect, if any.
In steady conditions all vectors are independent of time. Therefore, the two equations governing the electric field, namely (5.1.7) and (5.1.8), are decoupled with respect to the two equations governing the magnetic field, namely (5.1.9) and (5.1.10) (see Section 2.2 and Section 2.3).
5.2 Poynting’s vector

Let Maxwell’s equations (5.1.7) and (5.1.9) be considered. By means of a vector identity (see A.13) one obtains

\[
\nabla \cdot (E \times H) = \nabla \cdot (\nabla \times E) - E \cdot (\nabla \times H) = -\frac{\partial B}{\partial t} - E \cdot \frac{\partial D}{\partial t} - \nabla \cdot J \quad (5.2.1)
\]

Referring to the specific energy in the electric and magnetic case and under the assumption of linear constitutive relationship, one has

\[
\frac{1}{2} \frac{\partial}{\partial t} (H \cdot B + E \cdot D) = \frac{1}{2} \left( H \cdot \frac{\partial B}{\partial t} + B \cdot \frac{\partial H}{\partial t} \right) + \frac{1}{2} \left( E \cdot \frac{\partial D}{\partial t} + D \cdot \frac{\partial E}{\partial t} \right)
\]

\[
= \frac{\partial B}{\partial t} + \frac{\partial D}{\partial t} \quad (5.2.2)
\]

Integrating (5.2.1) over \( \Omega \) and using Gauss’s theorem (see A.10), it results

\[
\int_{\Gamma} (E \times H) \cdot d\Gamma = -\frac{\partial}{\partial t} \int_{\Omega} \left( \frac{H \cdot B}{2} + \frac{E \cdot D}{2} \right) d\Omega - \int_{\Omega} E \cdot J d\Omega \quad (5.2.3)
\]

Vector

\[
\vec{S} = E \times H \quad (5.2.4)
\]

is called Poynting’s vector (Wm\(^{-2}\)).

According to (5.2.3), its flux out of a closed surface \( \Gamma \) is equal to (minus) the sum of the power of the electromagnetic field inside the domain \( \Omega \) and the power transferred to the current (Poynting’s theorem).
5.3 Maxwell’s equations in the frequency domain

The most important case of time-varying electromagnetic fields occurs when field sources, namely charge and current densities, vary with sinusoidal law. A given vector

\[ \nabla (x, y, z, t) = [V_{0x}(x, y, z) \cos(\omega t - \phi), V_{0y}(x, y, z) \cos(\omega t - \phi), V_{0z}(x, y, z) \cos(\omega t - \phi)] = \]

\[ = \nabla_0 \cos(\omega t - \phi) \] (5.3.1)

can be expressed as

\[ \nabla (x, y, z, t) = [V_{0x}(x, y, z) \text{Re} e^{j(\omega t - \phi)}, V_{0y}(x, y, z) \text{Re} e^{j(\omega t - \phi)}, V_{0z}(x, y, z) \text{Re} e^{j(\omega t - \phi)}] = \]

\[ = \nabla_0 \text{Re} e^{j(\omega t - \phi)} \] (5.3.2)

The algebraic quantity \( \nabla = \nabla_0 e^{-j\phi} \) (phasor) represents the vector \( \nabla(x, y, z, t) \) in a unique way; moreover, in the frequency domain, since \( \frac{d}{dt} \cos(\omega t) = \omega \cos(\omega t + \frac{\pi}{2}) \), the differential operator \( \frac{\partial}{\partial t} \) is transformed into the complex operator \( j\omega \).

Consequently, Maxwell’s equation (5.1.7), (5.1.8), (5.1.9) and (5.1.10) are transformed as follows:

\[ \nabla \times E = -j\omega B \] (5.3.3)

\[ \nabla \cdot B = 0 \] (5.3.4)

\[ \nabla \times H = J + j\omega D \] (5.3.5)

\[ \nabla \cdot D = \rho \] (5.3.6)

The latter equations are referred to as Helmholtz’s equations and are valid at sinusoidal steady state for frequency \( f = \frac{\omega}{2\pi} \) (field equations in the frequency domain).

It should be remarked that field quantities in the latter equations are the phasors corresponding to the associated time functions; by definition, the amplitude of the phasor is the maximum value of the corresponding time function.
Considering the constitutive equations, in a non-conducting region free of spatial charges \((\rho = 0)\) and impressed currents, from vector identity (A.12), taking into account (5.3.6) one has

\[
\nabla \times \nabla \times \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E} \tag{5.3.7}
\]

From (5.3.3) and (5.3.5), if \(\mu\) is a constant and \(\sigma = 0, J_0 = 0\), it follows

\[
\nabla \times \nabla \times \mathbf{E} = \nabla \times (-j_0 \mathbf{B}) = -j_0 \nabla \times \mathbf{B} =
\]

\[
= -j_0 \mu \nabla \times \mathbf{H} = -j_0 \mu (j_0 \mathbf{D}) = \omega^2 \mu \varepsilon \mathbf{E} \tag{5.3.8}
\]

Comparing (5.3.7) and (5.3.8), Helmholtz's equation of electric field results

\[
\nabla^2 \mathbf{E} = -\omega^2 \mu \varepsilon \mathbf{E} = k^2 \mathbf{E} \tag{5.3.9}
\]

with \(k = j_0 \sqrt{\mu \varepsilon}\).

If the same procedure is applied to field \(\mathbf{H}\), one obtains

\[
\nabla^2 \mathbf{H} = -\omega^2 \mu \varepsilon \mathbf{H} = k^2 \mathbf{H} \tag{5.3.10}
\]

At sinusoidal steady state, the Poynting's vector (phasor) resulting from the time average of (5.2.4), considering the root-mean-square value of each vector, is

\[
\mathbf{S} = \frac{\mathbf{E} \times \mathbf{H}^*}{2} \tag{5.3.11}
\]

where the star denotes the conjugate phasor.

Referring to a volume \(\Omega\) with boundary \(\Gamma\), (5.2.3) takes the form

\[
\int_{\Gamma} \left( \frac{\mathbf{E} \times \mathbf{H}^*}{2} \right) \cdot \mathbf{n} \, d\Gamma = -2j_0 \omega \int_{\Omega} \left( \frac{\mathbf{H} \cdot \mathbf{B}^*}{4} + \frac{\mathbf{E} \cdot \mathbf{D}^*}{4} \right) \, d\Omega - \int_{\Omega} \frac{\mathbf{E} \cdot \mathbf{J}^*}{2} \, d\Omega \tag{5.3.12}
\]
5.4 Plane waves in an infinite domain

Let a simply-connected unbounded domain, filled in by a perfectly insulating medium ($\rho = 0$, $\sigma = 0$), be considered. For the sake of simplicity, let a time-harmonic electric field $E_0 \cos \omega t$ have only a non-zero component in the y-direction and vary only in the x direction (Fig. 5.1).

![Fig. 5.1 – Travelling plane electromagnetic wave.](image-url)

The Helmholtz’s equation (5.3.9) reduces to

$$\frac{\partial^2 E}{\partial x^2} = -\omega^2 \mu \varepsilon E$$  \hspace{1cm} (5.4.1)

It can be easily proven that the complex function

$$E = E_0 e^{j\omega \sqrt{\mu \varepsilon} \left( x - \frac{1}{\sqrt{\mu \varepsilon}} t \right)}$$  \hspace{1cm} (5.4.2)

with $E_0$ phasor of the given electric field, is a solution of (5.4.1).

In the time domain it results

$$E(x, t) = E_0 \cos \left[ \frac{\omega}{u} (x - ut) \right]$$  \hspace{1cm} (5.4.3)

with $u = \frac{1}{\sqrt{\mu \varepsilon}}$ (ms$^{-1}$). It can be verified that also

$$E(x, t) = E_0 \cos \left[ \frac{\omega}{u} (x + ut) \right]$$  \hspace{1cm} (5.4.4)
once transformed in its complex form \( \vec{E} = E_0 e^{j \omega \sqrt{\mu \varepsilon} \left(x + \frac{1}{\sqrt{\mu \varepsilon}} t\right)} \) is a solution of (5.4.1).

From the physical standpoint, (5.4.3) and (5.4.4) represent harmonic waves travelling with velocity \( u \) in positive and negative \( x \)-direction, respectively. Owing to (5.3.3) a time-harmonic field \( \vec{B} \) is associated to \( \vec{E} \).

In the frequency domain it results

\[
\vec{B} = \begin{pmatrix} 0, \ 0, \ \frac{1}{j \omega} \frac{\partial \vec{E}}{\partial x} \end{pmatrix} = \begin{pmatrix} 0, \ 0, \ \sqrt{\mu \varepsilon} \ \vec{E} \end{pmatrix}
\]  \hspace{1cm} (5.4.5)

In the time domain one obtains:

\[
B(x, t) = \frac{E_0}{u} \cos \left[ \frac{\omega}{u} (x \pm u t) \right]
\]  \hspace{1cm} (5.4.6)

From (5.4.3),(5.4.4) and (5.4.5), it results that the couple of vectors \( \vec{E}, \vec{B} \) defined above is a plane wave; \( \vec{E} \) and \( \vec{B} \) are orthogonal vectors; the ratio of electric field intensity to induction field intensity is equal to the velocity \( u \) of propagation in the dielectric medium.

Moreover, the Poynting’s vector \( \vec{S} \) in the time domain results

\[
\vec{S} = \vec{E} \times \vec{H} = \frac{E_0^2}{u \mu} \left[ 1 + \cos \left( \frac{2 \omega}{u} (x \pm u t) \right) \right] i_x
\]  \hspace{1cm} (5.4.7)

Therefore, the direction of propagation of the plane wave is orthogonal to both electric and magnetic field (transverse electromagnetic wave, TEM).
5.5 Wave and diffusion equations in terms of vectors $\vec{E}$ and $\vec{H}$

Considering the constitutive relations (2.2.1), (2.3.1), (2.4.1), Maxwell's equations (5.1.7) and (5.1.9) become, in terms of fields $\vec{E}$ and $\vec{H}$,

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad (5.5.1)$$

$$\nabla \times \vec{H} = \vec{J}_0 + \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \quad (5.5.2)$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} \quad (5.5.3)$$

$$\nabla \cdot \vec{H} = 0 \quad (5.5.4)$$

where $\vec{J}_0$ is the impressed current density.

From (5.5.1) one has

$$\nabla \times \nabla \times \vec{E} = -\frac{\partial}{\partial t} \left( \nabla \times \mu \vec{H} \right) \quad (5.5.5)$$

Since (see A.12)

$$\nabla \times \nabla \times \vec{E} = \nabla \left( \nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} \quad (5.5.6)$$

taking into account that, in the absence of free charges (i.e. $\rho = 0$), if $\varepsilon$ is a constant $\nabla \cdot \vec{E} = \nabla \cdot \frac{\vec{D}}{\varepsilon} = 0$, one obtains

$$-\nabla^2 \vec{E} = -\frac{\partial}{\partial t} \left( \nabla \times \mu \vec{H} \right) \quad (5.5.7)$$

Then, for a homogeneous medium it results

$$\nabla^2 \vec{E} = \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu \sigma \frac{\partial \vec{E}}{\partial t} + \mu \frac{\partial \vec{J}_0}{\partial t} \quad (5.5.8)$$

namely, the equation governing electric field $\vec{E}$; if $\frac{\partial \vec{J}_0}{\partial t} = 0$, the homogeneous wave equation is obtained.

Similarly, it can be proven that for field $\vec{H}$ the following equation holds
If in (5.5.2) the displacement current density \( \varepsilon \frac{\partial E}{\partial t} \) can be neglected, then equations (5.5.8) and (5.5.9) become

\[
\nabla^2 H = \mu \varepsilon \frac{\partial^2 H}{\partial t^2} + \mu \sigma \frac{\partial H}{\partial t} - \nabla \times J_0 \tag{5.5.10}
\]

and

\[
\nabla^2 E = \mu \sigma \frac{\partial E}{\partial t} + \mu \frac{\partial J_0}{\partial t} \tag{5.5.11}
\]

respectively; they are the differential equations governing the electromagnetic field under quasi-static conditions (diffusion equations).

In turn, by taking the divergence of both sides of (5.5.2) and considering (A.8), the equation of charge relaxation follows

\[
\nabla \cdot \left( \sigma E + \varepsilon \frac{\partial E}{\partial t} \right) = -\nabla \cdot J_0 \tag{5.5.12}
\]

where the driving term is due to the impressed current density. It can be remarked that (5.5.12) states the current density balance in a dissipative dielectric medium, characterised by both conductivity \( \sigma \) and permittivity \( \varepsilon \).

In the frequency domain, (5.5.12) transforms as

\[
\nabla \cdot \left( \sigma E + j \omega \varepsilon E \right) = -\nabla \cdot J_0 \tag{5.5.13}
\]

where the complex conductivity \( \sigma + j \omega \varepsilon \) appears; in (5.5.13) \( E \) and \( J_0 \) are the phasors corresponding to the associated time functions.
5.6 Wave and diffusion equations in terms of scalar and vector potentials

In a simply connected domain \( \Omega \) filled in by a linear and homogeneous medium, the magnetic vector potential \( \overline{A} \) (Wb m\(^{-1}\)) is defined by the equation (see 2.3.19)

\[
\overline{B} = \nabla \times \overline{A}
\]  

(5.6.1)

associated to a suitable gauge condition to be specified later on.

By means of (5.1.7) one has

\[
\nabla \times \left( \overline{E} + \frac{\partial \overline{A}}{\partial t} \right) = 0
\]

(5.6.2)

This means that the vector in brackets can be expressed as the gradient of a scalar potential \( \varphi \) (V)

\[
\overline{E} + \frac{\partial \overline{A}}{\partial t} = -\nabla \varphi
\]

(5.6.3)

Hence

\[
\overline{E} = -\nabla \varphi - \frac{\partial \overline{A}}{\partial t}
\]

(5.6.4)

Substituting (5.6.4) into (5.5.2) one obtains

\[
\nabla \times \overline{H} = \overline{J}_0 - \sigma \nabla \varphi - \sigma \frac{\partial \overline{A}}{\partial t} + \varepsilon \frac{\partial}{\partial t} \nabla \varphi - \varepsilon \frac{\partial^2 \overline{A}}{\partial t^2}
\]

(5.6.5)

From (5.6.1) one has

\[
\nabla \times \overline{H} = \nabla \times \mu^{-1} \nabla \times \overline{A}
\]

(5.6.6)

and

\[
\nabla \times \nabla \times \overline{A} + \mu \varepsilon \frac{\partial \varphi}{\partial t} + \mu \varepsilon \frac{\partial^2 \overline{A}}{\partial t^2} = \mu \left( \overline{J}_0 - \sigma \nabla \varphi - \sigma \frac{\partial \overline{A}}{\partial t} \right)
\]

(5.6.7)

In the case of a current-free and charge-free ideal dielectric region (\( J_0 = 0 \), \( \rho = 0 \) and \( \sigma = 0 \)) it results

\[
\nabla \times \nabla \times \overline{A} + \mu \varepsilon \frac{\partial \varphi}{\partial t} + \mu \varepsilon \frac{\partial^2 \overline{A}}{\partial t^2} = 0
\]

(5.6.8)

On the other hand, substituting (5.6.3) into (5.5.3) gives

\[
-\nabla^2 \varphi - \frac{\partial}{\partial t} (\nabla \cdot \overline{A}) = 0
\]

(5.6.9)
Equations (5.6.7) and (5.6.9) represent the link between the two potentials. Taking into account that (see A.12)

$$\nabla \times \nabla \times A = -\nabla^2 A + \nabla (\nabla \cdot A) \tag{5.6.10}$$

and by substituting this expression into (5.6.8) one has

$$-\nabla^2 A + \nabla (\nabla \cdot A) + \mu_0 \nabla \frac{\partial \phi}{\partial t} + \mu_0 \frac{\partial^2 A}{\partial t^2} = 0 \tag{5.6.11}$$

or

$$-\nabla^2 A + \nabla (\nabla \cdot A + \mu_0 \frac{\partial \phi}{\partial t}) + \mu_0 \frac{\partial^2 A}{\partial t^2} = 0 \tag{5.6.12}$$

If the Lorentz's gauge

$$\nabla \cdot A + \mu_0 \frac{\partial \phi}{\partial t} = 0 \tag{5.6.13}$$

is imposed, then from (5.6.12) one obtains

$$-\nabla^2 A + \mu_0 \frac{\partial^2 A}{\partial t^2} = 0 \tag{5.6.14}$$

which is the wave equation for the magnetic vector potential $A$, subject to boundary and intial conditions. After determining $A$, following (5.6.13), $\phi$ is given by

$$\phi(t) = \phi_0 - \frac{1}{\mu_0} \int_0^1 \nabla \cdot A(t') dt' \tag{5.6.15}$$

with $\phi_0$ to be determined.

Alternatively, imposing gauge (5.6.13) to equation (5.6.9), the wave equation for the electric scalar potential is obtained

$$-\nabla^2 \phi + \mu_0 \frac{\partial^2 \phi}{\partial t^2} = 0 \tag{5.6.16}$$

After determining $\phi$, $A$ can be recovered.

In the case current $J_0$ and charge $\rho$ are present, (5.6.14) and (5.6.16) become

$$-\nabla^2 A + \mu_0 \frac{\partial^2 A}{\partial t^2} = \mu J_0 \tag{5.6.17}$$
\(- \nabla^2 \varphi + \mu \varepsilon \frac{\partial^2 \varphi}{\partial t^2} = \frac{\rho}{\varepsilon} \)  
(5.6.18)

respectively.

In a three-dimensional unbounded domain \( \Omega \), their particular solutions are (see 2.1.48 and 2.1.47)

\[
\overline{A} = \int_{\Omega} \frac{\mu \overline{J}_0}{4\pi r} \, d\Omega \\
\varphi = \int_{\Omega} \frac{\rho'}{4\pi r} \, d\Omega
\]
(5.6.19)  
(5.6.20)

where the values \( \overline{J}_0 \) and \( \rho' \) are taken at an earlier time \( t' = t - r\sqrt{\mu \varepsilon} \) with respect to the time \( t \) at which \( \overline{A} \) and \( \varphi \) are observed. The latter two potentials are therefore called retarded potentials. Additionally, it can be noted that \( \overline{A} \) depends only on \( \overline{J}_0 \) and \( \varphi \) depends only on \( \rho \). This dependence is, except for the correspondence of time, the same as in magnetostatics and electrostatics, respectively.

In the case of a conductor (\( \rho = 0 \), \( \sigma \neq 0 \)), by imposing the following gauge

\[
\nabla \cdot \overline{A} + \mu \varepsilon \frac{\partial \varphi}{\partial t} + \mu \sigma \varphi = 0
\]
(5.6.21)

from (5.6.7) and (5.6.10) it follows

\[- \nabla^2 \overline{A} + \left( \nabla \cdot \overline{A} + \mu \varepsilon \frac{\partial \varphi}{\partial t} + \mu \sigma \varphi \right) + \mu \varepsilon \frac{\partial^2 \overline{A}}{\partial t^2} + \mu \sigma \frac{\partial \overline{A}}{\partial t} = \mu \overline{J}_0
\]
(5.6.22)

or

\[- \nabla^2 \overline{A} + \mu \varepsilon \frac{\partial^2 \overline{A}}{\partial t^2} + \mu \sigma \frac{\partial \overline{A}}{\partial t} = \mu \overline{J}_0
\]
(5.6.23)

After determining \( \overline{A} \) and so \( \nabla \cdot \overline{A} \), \( \varphi \) can be recovered from (5.6.21).
5.7 Electromagnetic field radiated by an oscillating dipole

Let a point charge \( q(t) = q \sin(\omega t) \) oscillate with angular frequency \( \omega \) along an element \( d\ell \) of line \( \ell \) in a three-dimensional domain, so that the resulting current is \( i = \omega q \cos \omega t \). If line \( \ell \) is coincident with the z axis, in the frequency domain the phasor of the elementary vector potential (see 5.6.19) can be expressed as

\[
d\vec{A} = \frac{\mu_0}{4\pi} \frac{I}{r} e^{-j\omega r/c} d\ell \hat{z}
\]  

(5.7.1)

where \( I \) is the phasor of current \( i \), \( r \) is the distance between field point and source point, \( c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \) is the velocity of the electromagnetic wave in free space and the operator \( e^{-j\omega r/c} \) accounts for the phase delay of \( d\vec{A} \) with respect to \( \vec{I} \). Assuming spherical coordinates with origin at the gravity centre of the dipole (Fig. 5.2), the components of vector potential are

\[
d\vec{A}_r = d\vec{A} \cos \vartheta
\]

\[
d\vec{A}_\vartheta = -d\vec{A} \sin \vartheta
\]

\[
d\vec{A}_\varphi = 0
\]

(5.7.2)

Since \( \mu_0 \hat{H} = \nabla \times \vec{A} \), the components of the elementary magnetic field in the frequency domain are (see A.21-A.23)

\[
d\vec{H}_r = d\vec{H}_\vartheta = 0
\]

\[
d\vec{H}_\varphi = \frac{I}{4\pi} \frac{\sin \vartheta d\ell}{r^2} \left( 1 + j\frac{\omega r}{c} \right) e^{-j\omega r/c}
\]

(5.7.3)

Thanks to (5.7.3), it can be noted that lines of magnetic fields are circular and are located on planes normal to the direction of z axis. According to the Lorentz’s gauge (5.6.13), the elementary scalar potential associated to vector potential is

\[
d\phi = \frac{j \omega}{c} \nabla \cdot (d\vec{A})
\]

(5.7.4)
Considering (5.6.4) and (5.7.4), the relationship between potentials and electric field

\[ d\vec{E} = -j\omega d\vec{A} - \nabla \vec{\phi} \]  \hspace{1cm} (5.7.5)

becomes

\[ d\vec{E} = -j\omega d\vec{A} - j\frac{e^2}{\omega} \nabla (\nabla \cdot d\vec{A}) \]  \hspace{1cm} (5.7.6)

After (5.7.2), (A.19) and (A.24), the components of the elementary electric field follow

\[ d\vec{E}_r = -j\frac{2\hat{r} \cos \theta d\ell}{4\pi \varepsilon_0 r^3} \left( 1 + j\frac{\omega r}{c} \right) e^{-\frac{\omega r}{c}} \]

\[ d\vec{E}_\theta = -j\frac{\hat{\theta} \sin \theta d\ell}{4\pi \varepsilon_0 r^3} \left[ 1 + j\frac{\omega r}{c} - \left( \frac{\omega r}{c} \right)^2 \right] e^{-\frac{\omega r}{c}} \]

\[ d\vec{E}_\phi = 0 \] \hspace{1cm} (5.7.7)

The situation is represented in Fig. 5.2.

![Fig. 5.2 - Field radiated at point P by an oscillating dipole.](image_url)

It is interesting to consider the approximated expressions of field components near the oscillating dipole and far from it, respectively.
Under the approximation $\frac{\omega r}{c} << 1$ of near field, the field components become

$$d\vec{H}_\phi = \frac{\tilde{I} \sin \vartheta d\ell}{4\pi r^2}$$  \hspace{1cm} (5.7.8)$$

$$d\vec{E}_r = -j \frac{\tilde{I} \cos \vartheta d\ell}{4\pi\varepsilon_0 r^3 \omega}$$  \hspace{1cm} (5.7.9)$$

$$d\vec{E}_\vartheta = -j \frac{\tilde{I} \sin \vartheta d\ell}{4\pi\varepsilon_0 r^3 \omega}$$  \hspace{1cm} (5.7.10)$$

It can be noted that the magnetic field scales as $\frac{1}{r^2}$ following the Laplace’s law of the elementary action valid for a steady current (see 3.1.79); in turn, the electric field scales as $\frac{1}{r^3}$ according to the static field of a dipole (see Section 2.2.6).

Conversely, under the approximation $\frac{\omega r}{c} >> 1$ of far field, the field components become

$$d\vec{H}_\vartheta = \frac{j \tilde{I} \sin \vartheta d\ell}{4\pi c} \left( \frac{\omega}{r} \right) e^{-j \frac{\omega t}{c}}$$  \hspace{1cm} (5.7.11)$$

$$d\vec{E}_\vartheta = \frac{j \tilde{I} \sin \vartheta d\ell}{4\pi\varepsilon_0 c^2} \left( \frac{\omega}{r} \right) e^{-j \frac{\omega t}{c}}$$  \hspace{1cm} (5.7.12)$$

The component $d\vec{E}_r$ can be neglected with respect to $d\vec{E}_\vartheta$, apart from points in which $|\sin \vartheta| << 1$. It is important to note that electric and magnetic fields are orthogonal, in phase and tangent to the sphere of radius $r$; consequently, the Poynting’s vector has a radial direction only and results

$$d\vec{S} = \frac{d\vec{E}_\vartheta \times d\vec{H}_\vartheta}{2} = \frac{\tilde{I}^2 \sin^2 \vartheta \left( \frac{d\ell}{\ell} \right)^2 \left( \frac{\omega}{r} \right)^2}{16\pi^2\varepsilon_0 c^3}$$  \hspace{1cm} (5.7.13)$$

where $\tilde{I}$ is the root-mean-square value of current.

It comes out that the power radiated by the dipole is maximum for $\vartheta = \frac{\pi}{2}$ (equatorial plane) and zero for $\vartheta = 0$ (z axis); furthermore, the average
power flowing through a spherical surface is independent of its radius. Finally, the amplitude of fields depends on $\frac{\omega}{r}$; therefore, to make a long-distance transmission, it is necessary to increase the source frequency.
5.8 Diffusion equation in terms of dual potentials

Let a linear homogeneous isotropic medium, characterized by conductivity \( \sigma \), permeability \( \mu \) and permittivity \( \varepsilon \) be considered, where an impressed current \( J_0 \) is present, the time variations of which are small, i.e., if time harmonic variations occur, the angular frequency is much lower than \( \frac{\sigma}{\varepsilon} \).

Then, displacement current density \( \frac{\partial D}{\partial t} \) may be neglected with respect to impressed \( \bar{J}_0 \) and induced \( \sigma E \) current densities (quasi-static approximation). In this case, Maxwell’s equations reduce to

\[
\nabla \times E = -\frac{\partial B}{\partial t} \tag{5.8.1}
\]
\[
\nabla \cdot D = 0 \tag{5.8.2}
\]
\[
\nabla \times H = \bar{J} = \bar{J}_0 + \sigma E \tag{5.8.3}
\]
\[
\nabla \cdot B = 0 \tag{5.8.4}
\]

along with the constitutive relations (5.1.2) and (2.3.1).

Given appropriate boundary and initial conditions, vectors \( \bar{H} \) (or \( \bar{B} \)) and \( \bar{E} \) (or \( \bar{J} \) and \( \bar{D} \)) are uniquely defined (see Section 2.1.2).

This is a special case of Section 5.1 and is particularly important in low-frequency applications (eddy current problem).

The electromagnetic field can be also described in terms of potentials in two different ways.

According to the \( \bar{A} \)-\( \phi \) method (see Section 5.6) a magnetic vector potential \( \bar{A} \) (Wb m\(^{-1}\)) is introduced by (5.6.1); moreover, an electric scalar potential \( \phi \) is defined according to (5.6.4).

In order to specify \( \bar{A} \) uniquely, a further condition must be introduced: this may be the Coulomb’s gauge (2.3.20) or the Lorentz’s gauge (5.6.13).

This way \( \bar{E} \) and \( \bar{H} \) can be expressed by means of two potentials (see Section 2.1.4), namely \( \bar{A} \) and \( \phi \).

From (5.8.3) taking into account (5.6.1) and (5.6.4) one has

\[
\nabla \times \mu^{-1} \nabla \times \bar{A} = \bar{J}_0 - \sigma \frac{\partial \bar{A}}{\partial t} - \sigma \nabla \phi \tag{5.8.5}
\]

From (5.1.5), taking into account (5.6.4), it follows
Equations (5.8.5) and (5.8.6) with appropriate boundary and initial conditions solve the electromagnetic problem in terms of $\vec{A}$ and $\phi$. In a region where $\sigma=0$ (eddy-current free) the latter reduce to the classical equations of magnetostatics (see Section 2.3.1). On the other hand, (5.8.5) is a special case of (5.6.7).

Moreover, imposing the gauge $\nabla \cdot \vec{A} + \mu \phi = 0$, from (5.8.5) one obtains

$$\vec{J}_0 = \mu \sigma \frac{\partial \vec{A}}{\partial t} - \sigma \nabla \phi$$

that represents the diffusion equation in terms of vector potential; it is an approximation of equation (5.6.23) in the quasi-static state. After determining $\vec{A}$, scalar potential $\phi = - (\mu \sigma)^{-1} \nabla \cdot \vec{A}$ is recovered.

Alternatively, following the $\vec{T} - \Omega$ method, in regions free of impressed current ($J_0 = 0$) an electric vector potential $T$ (A m$^{-1}$) can be defined as

$$\nabla \times \vec{T} = \vec{J}$$

Comparing (5.8.8) and (5.8.3) it turns out that $\vec{H}$ and $\vec{T}$, which have the same curl, must differ by the gradient of a function $\Omega$ (A) (magnetic scalar potential)

$$\vec{H} = \vec{T} - \nabla \Omega$$

The electric and magnetic vectors, $\vec{J}$ and $\vec{H}$, have been so expressed in terms of two potentials. In order to define $\vec{T}$ uniquely, a gauge must be introduced. The equation governing the electromagnetic field can be now expressed in terms of $\vec{T}$ and $\Omega$. In fact, from (5.8.3) taking the curl of both members and taking into account (5.8.1) and (5.8.9), one has

$$\nabla \times \left( \sigma^{-1} \nabla \times \vec{T} \right) = \nabla \times \sigma^{-1} \vec{J}_0 - \frac{\partial \mu}{\partial t} \nabla \Omega$$

and from (5.8.4)

$$\nabla \cdot \mu (\vec{T} - \nabla \Omega) = 0$$

In regions where $\sigma = 0$ one has $\vec{J} = 0$ and therefore, from (5.8.4), $\nabla \times \vec{T} = 0$. 
Moreover, imposing the gauge \( \nabla \cdot \mathbf{T} = \mu \sigma \frac{\partial \Omega}{\partial t} \), from (5.8.10) and (5.8.11) one obtains two independent equations for \( T \) and \( \Omega \), namely

\[
\nabla^2 T - \mu \sigma \frac{\partial T}{\partial t} = -\nabla \times \mathbf{J}_0 
\]

(5.8.12)

and

\[
\nabla^2 \Omega - \mu \sigma \frac{\partial \Omega}{\partial t} = 0 
\]

(5.8.13)

subject to appropriate boundary conditions. They are

\[
\mathbf{n} \times \mathbf{T} = 0 \quad , \quad \Omega = 0 
\]

(5.8.14)

or

\[
\mathbf{n} \cdot \mathbf{T} = 0 \quad , \quad \frac{\partial \Omega}{\partial n} = 0 
\]

(5.8.15)

if the boundary is normal to a flux line (i.e. \( \mathbf{n} \times \mathbf{B} = 0 \)) or it is parallel to a flux line (i.e. \( \mathbf{n} \cdot \mathbf{B} = 0 \)), respectively.

After determining \( \mathbf{T} \), \( \Omega \) is given by

\[
\Omega(t) = \Omega_0 + (\mu \sigma)^{-1} \int_0^t \nabla \cdot \mathbf{T}(t')dt' 
\]

(5.8.16)

with \( \Omega_0 \) to be determined.