A new locking-free equilibrium mixed element for plane elasticity with continuous displacement interpolation

R. Nascimbene, P. Venini *

Department of Structural Mechanics, University of Pavia, Via Ferrata 1, I-27100 Pavia, Italy

Received 24 January 2001; received in revised form 29 June 2001; accepted 7 September 2001

Abstract

With reference to plane elastic problems, we first show that using the Johnson and Mercier (JM) element [C. Johnson, B. Mercier, Numer. Math. 30 (1978) 103] for the stresses and an $H^1$ piecewise linear interpolation for the displacements leads to a severe locking when the Lamé coefficient $\lambda$ tends to infinity, i.e. in the case of an incompressible material. Motivated by the need to have a continuous displacement field that emerges in several engineering applications, we use an eigenvalue argument to prove that locking is due to the presence of the (element-average) pressure among the degrees of freedom. We therefore relax the incompressibility constraint by introducing a quadrilateral macroelement basically consisting of two adjacent JM triangles sharing the same average pressure, in a sense switching from a (locking) $T_1P_0$ approximation to a locking-free $Q_1P_0$. Numerical tests on the pathological Cook cantilever are performed to assess the validity of the proposed approach in which comparisons between the newly devised element and the original JM element with $L^2$ displacements are presented. © 2001 Published by Elsevier Science B.V.

Keywords: Mixed finite element method; Incompressible elasticity; Locking; Elasticity; Finite element and matrix methods; Incompressible and near incompressible media

1. Introduction and motivation

Nearly all mixed methods of practical use for elastic and elastoplastic problems find their variational justification in Hellinger–Reissner or Hu-Washizu principles [7]. The (broad) family of Hu-Washizu variational principles derives from a three-field functional whereby stresses, strains and displacements are independently interpolated. Technically speaking, however, displacements and strains are the unknowns and the stress plays the role of a Lagrangian multiplier. As to the Hellinger–Reissner approach, there exist actually two different versions according to whether one seeks for a regular displacement field and a discontinuous stress field or vice versa. In both cases, though, the strain is ruled out and does not appear as an independent unknown. The engineering literature has widely investigated applications of the Hu-Washizu
principle as well as of the Hellinger–Reissner approach with regular displacements. Conversely, very few analyses of elastic and elasto-plastic problems have been proposed that make use of the Hellinger–Reissner principle with regular stresses. A (partial) motivation may be found in the difficulty with which the inf–sup condition may be enforced in this latter case [4,10]. In fact, usual polynomial finite elements fail in approximating properly the anisotropic space $H(\text{div})$ which is the natural functional space for the stress field. The lack of engineering applications does not match with the variety of theoretical strategies proposed in the mathematical literature that make use of the regular-stress Hellinger–Reissner approach. Roughly speaking one has at disposal two approaches that lead to FEM approximations that satisfy the inf–sup condition, namely [10]

1. to impose the symmetry of the stress tensor in weak form via a Lagrange multiplier rather than doing it a priori (as nearly always done) [2];
2. to replace polynomial shape functions, e.g. by means of composite elements [12,15].

Nonsymmetric stress tensors were exploited in [1–3,9,11,17] among others where the interested reader may find additional details on this strategy.

We will be dealing hereafter with the second available technique, moving from the Johnson–Mercier (JM) [15] composite element in compressible and incompressible elasticity. The motivation of our analysis is twofold. On the one hand, the mathematical literature, only in minimal part outlined herein, puts at disposal efficient schemes along with powerful error estimates and convergence rate studies that are often not available when using “engineering” approaches. Furthermore, from a physical point of view, an accurate interpolation for the stresses is highly desirable in most applications where, however, discontinuous displacements can seldom be tolerated. We therefore propose a new element enjoying the advantages of the JM element but characterized by continuous displacement fields at the same time. The main advantages of using continuous displacement interpolations are the following two:

1. from a computational viewpoint, the total number of degrees-of-freedom is considerably reduced,
2. the acceptance of the element by practitioners may be easier since discontinuous displacements might be perceived as leading to a non-physical fractured deformed shape.

For completeness sake, we should mention that smoothing techniques are available for regularizing a-posteriori discontinuous displacement patterns so as to comply with the second requirement above.

The paper outline is as follows. We first present the variational formulation of the problem in its continuous and discrete versions, both making use of the regular-stress Hellinger–Reissner variational principle. We then proceed with the introduction of the JM element commenting on the results of a few numerical simulations that are in excellent agreement with the relevant theoretical error estimates [15]. By simply modifying the displacement approximation from an $L^2$ to an $H^1$ piecewise linear one, we then show that a severe locking appears for (almost and) incompressible materials. The body of the paper is then the proposal of a new mixed finite element that incorporates all the good features of the JM one allowing the analysis of incompressible materials at the same time. Numerical tests are eventually discussed to complete our investigation.

2. The elastic problem

We consider the static behavior of an elastic body which occupies a bounded domain $\Omega \subset \mathbb{R}^2$ with a sufficiently smooth boundary $\partial \Omega = \Gamma$. The system is initially at rest, undeformed and unstressed. The equations governing the linear elastic problem in two dimensions can be formulated as:
the translational equilibrium equations:
\[ \text{div} \sigma + f = 0 \quad \text{in } \Omega, \tag{1} \]
where \( \sigma \) is the stress tensor and \( f \) is the load per unit volume (rotational equilibrium make us able to say that \( \sigma = \sigma^T \));

- the elastic constitutive equation reads:
\[ \sigma = C \varepsilon, \tag{2} \]
in which \( \varepsilon \) is the linearized strain tensor and \( C \) is a fourth-order tensor of elastic coefficients enjoying the symmetry properties:
\[ C_{ijkl} = C_{jikl} = C_{klij}. \tag{3} \]
Furthermore, \( C \) is bounded, i.e.
\[ C_{ijkl} \in L^\infty(\Omega), \tag{4} \]
and pointwise stable, i.e. there exists a constant \( c_0 > 0 \) such that
\[ C_{ijkl}(x)\varepsilon_{ij}\varepsilon_{kl} \geq c_0 \varepsilon_{ij}\varepsilon_{ij} \quad \forall \varepsilon \in M^2, \text{ a.e. in } \Omega, \tag{5} \]
where \( M^2 \) is the space of symmetric second order tensors in \( \mathbb{R}^2 \). It should be clear from these definitions that pointwise stability implies, but is not implied by, strong ellipticity. In the case of isotropic elasticity, the (inverse of the) constitutive law (2) reads
\[ \varepsilon(u) = \lambda \text{tr}(\sigma)\delta + \mu \sigma \quad \text{in } \Omega, \tag{6} \]
where the unit second-order tensor is denoted by \( \delta \), and \( \lambda \) and \( \mu \) are the material Lame’s constants;

- under the small displacement gradients assumption, the compatibility condition between the displacement vector \( u \) and the strain tensor \( \varepsilon \) reads
\[ \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T); \tag{7} \]

- finally the theoretical derivations are worked out with homogeneous boundary conditions, i.e.
\[ u = 0 \text{ on } \Gamma, \tag{8} \]
even though numerical simulations will have general boundary conditions.

### 2.1. Variational formulation

The following functional spaces are of interest
\[ V = [H^1_0(\Omega)]^2, \quad \Sigma = \{ \varepsilon : \tau_{ij} \in L^2(\Omega), \tau_{ij} = \tau_{ji} \}, \quad W = [L^2(\Omega)]^2, \]
and
\[ H = H(\text{div}; \Omega) = \{ \varepsilon : \tau_{ij} = \tau_{ji}, \tau_{ij} \in L^2(\Omega), \text{div} \varepsilon \in W; i, j = 1, 2 \}, \]
where \( \text{div} \varepsilon \) is the vector with components \( \sum_j \partial \tau_{ij}/\partial x_j \). As usual,
\[ L^2(\Omega) = \left\{ v | \int_{\Omega} |v|^2 \, dx = ||v||_{L^2(\Omega)}^2 < +\infty \right\}, \]
is the space of square integrable functions on \( \Omega \). The space \( W \) has the standard \( L^2 \)-product norm, while \( H \) is endowed with the graph norm.
that makes \( H(\text{div}; \Omega) \) an Hilbert space. Mixed finite-element methods in elasticity originate from the classical Hellinger–Reissner principle that reads: find \((\sigma, u) \in \Sigma \times V\) such that
\[
\int_{\Omega} \sigma : C^{-1} \tau \, dx - \int_{\Omega} \varepsilon(u) : \tau \, dx = 0 \quad \forall \tau \in \Sigma,
\]
\[
\int_{\Omega} \varepsilon(\xi) : \xi \, dx = \int_{\Omega} g : \xi \, dx \quad \forall \xi \in V.
\]

The alternative version of the Hellinger–Reissner formulation for elasticity is the one in which the differentiability is transferred from the displacements to the stresses via integration by parts, i.e. find \((\sigma, u) \in H \times W\) such that
\[
\int_{\Omega} \sigma : C^{-1} \tau \, dx + \int_{\Omega} u \cdot \text{div} \tau \, dx = 0 \quad \forall \tau \in H(\text{div}; \Omega),
\]
\[
\int_{\Omega} \varepsilon(\xi) : \xi \, dx - \int_{\Omega} \text{div}(\sigma) \, dx = -\int_{\Omega} g : \xi \, dx \quad \forall \xi \in W(\Omega).
\]

In the case of isotropic elasticity, thanks to (6), one may write: find \((\sigma, u) \in H \times W\) such that
\[
a(\sigma, \tau) + b(\tau, u) = 0 \quad \forall \tau \in H(\text{div}; \Omega),
\]
\[
b(\sigma, \tau) + (g, \tau) = 0 \quad \forall \tau \in W(\Omega),
\]
where \((\cdot, \cdot)\) denote the scalar product in \(W\) and
\[
a(\sigma, \tau) = \int_{\Omega} \left( \frac{1}{2\mu} \sigma^p : \tau^p + \frac{1}{\lambda + \mu} \text{tr}(\sigma) \text{tr}(\tau) \right) \, dx,
\]
\[
b(\tau, u) = \int_{\Omega} \text{div}(\tau) \cdot u \, dx.
\]

One may then check that [10]
\[
\inf_{\tau \in W(\Omega)} \sup_{\xi \in H} \frac{b(\tau, \xi)}{\|\xi\|_H \|\tau\|_0} \geq c > 0,
\]
and that
\[
a(\tau, \tau) \geq \frac{1}{2(\lambda + \mu)} \|\tau\|_0^2.
\]

Moreover, using the symbol \(B\) to define the linear divergence operator from \(H(\text{div}; \Omega)\) to \(L^2(\Omega)\), we set:
\[
\text{Ker}B = \{\xi \in H | b(\xi, \xi) = 0 \ \forall \xi \in W\} = \{\xi \in H | \text{div} \xi = 0\},
\]
and one may prove that
\[
a(\tau, \tau) \geq c(\mu) \|\tau\|_H^2 \quad \forall \tau \in \text{Ker}B.
\]
Hence from (9), (17) and (19) one gets the well posedness of problem (14). The use of composite elements for the mixed formulation of elasticity problems was introduced by JM, see [10,15] for details. Here we introduce the JM element for triangles applied to the solution of Eqs. (12) and (13).

3. JM element

Going on now to finite element approximations of the above problem, we will assume for simplicity that the domain $\Omega$ is polygonal. Let $\mathcal{T}_h$ be a family of triangulations of $\Omega$, i.e.

$$\Omega = \bigcup_{T \in \mathcal{T}_h},$$

indexed by a parameter $h$ representing the maximum diameter of the elements $T$. We shall consider the case of triangular JM elements. Each $T \in \mathcal{T}_h$ will itself be divided into 3 subtriangles $T_i$, $i = 1, 2, 3$, see Fig. 1. We shall assume [15] that $\mathcal{T}_h$ is regular in the sense that all angles of the triangular elements $T \in \mathcal{T}_h$ are bounded away from zero and $\pi$ uniformly in $h$ and there is a positive constant $\alpha$ such that the length of any side of any $T \in \mathcal{T}_h$ is at least $\alpha h$. Now consider the space

$$\text{JM}(T) = \{ \varepsilon \in H(\text{div}; T), \quad \varepsilon |_{T_i} \in [P_1(T_i)]^4, \quad j = 1, 2, 3 \},$$

(21)

where $P_1(T_i)$ is the space of the polynomials of degree $\leq 1$ on $T_i$. Note in definition (21) that the condition $\varepsilon \in H(\text{div}; T)$ requires $\varepsilon \cdot n$ to be continuous across each subtriangle. An element $\varepsilon$ of $\text{JM}_1(T)$ is uniquely determined by the following 15 degrees of freedom [15]

$$\int_{e_i} (\varepsilon \cdot n) \cdot p \, ds \quad \forall p \in (P_1(e_i))^2, \quad i = 1, 2, 3,$$

(22)

$$\int_T \varepsilon : p \, dx \quad \forall p \in (P_0(T))^{2\times2},$$

(23)

where $e_i$ denote the edges of the element. Furthermore we define the finite dimensional subspaces of $H_h \subset H$ and $W_h \subset W$ by

$$H_h = \{ \varepsilon_h \in H : \frac{\partial \varepsilon_h}{\partial n} |_{\partial \Omega} \in \text{JM}(T), \quad \int_{\Omega} \text{tr } \varepsilon_h \, dx = 0 \},$$

$$W_h = \{ \varepsilon_h : \varepsilon_h |_{T} \in [P_1(T)]^2, \quad T \in \mathcal{T}_h \}.$$
Then with the choice of spaces $H_h$ and $W_h$ given above, it can be shown \cite{10} that problem (24) has a unique solution, and that, if $\bar{\sigma} \in [H^2(\Omega)]^{N_x \times 2}$, $\bar{u} \in [H^2(\Omega)]^2$ and $\lambda < \infty$ then the following estimates hold

$$\|\bar{\sigma} - \sigma_h\|_0 \leq C h^2,$$  

(25)

and

$$\|\bar{u} - u_h\|_0 \leq C h^2,$$  

(26)

where $\| \cdot \|_0$ denotes the product $L^2$ norm. When the material is incompressible, nothing is changed as far as the displacement and stress deviator estimates are concerned whereas a linear-in-$h$ estimate holds true for the pressure value.

4. Behavior in the presence of incompressible materials

4.1. Performance of the JM element

It is well known \cite{10,15} that no locking behavior is to be expected in the case of incompressible materials when the above approximations are adopted. This is in fact the case as shown in numerical simulations to come concerning the original JM formulation. From a numerical standpoint the absence of locking is not surprising if one recalls that displacements, i.e. Lagrange multipliers, are not only discontinuous but also linear over each element. The ratio

$$r = \frac{n_{eq}}{n_c},$$

described in \cite{14} among others, where $n_{eq}$ is the number of displacement equations and $n_c$ is the total number of incompressibility constraints, is then favorable and surely satisfying the heuristic no-locking condition $r \geq 2$. From an engineering viewpoint, however, the discontinuous displacement approximation may be too poor and unsatisfactory notwithstanding the estimate (26). Furthermore, the analysis of large-scale problems may be prevented by the presence of so many displacement degrees-of-freedom. If possible, one would like to have continuous displacements not giving up at the same time the appealing stress approximation of the JM element. Reported next are two proposals that move from a JM stress approximation coupled to a piecewise linear, globally $H^1$ interpolation for the displacements. The former is actually not successful and leads to locking but has the merit to open the way to the second formulation that is conversely locking-free and stable.

4.2. First trial: JM stresses and piecewise linear, globally $H^1$ displacements

As a first attempt, we consider an intermediate formulation in which displacements are still piecewise linear but globally continuous whereas nothing is changed as far as the stresses are concerned. It is worth observing that the above ratio $r$ is now more critical since the number of displacement equations is drastically reduced by the continuity requirement and therefore problems are to be expected. Attention is focused on the over-the-element average stress degrees-of-freedom of the JM element. They are usually written as \cite{10}

$$\int_T \sigma_{xx} \mathrm{d}\Omega, \quad \int_T \sigma_{yy} \mathrm{d}\Omega, \quad \int_T \sigma_{xy} \mathrm{d}\Omega.$$  

(27)
Obviously nothing is changed by switching to the averaged components of the stress deviator \( \mathbf{D} \), i.e.
\[
\int_T \frac{\sigma_{xx} - \sigma_{yy}}{2} \, d\Omega \quad \text{and} \quad \int_T \sigma_{xy} \, d\Omega,
\]
and the average in-plane normal stress
\[
\int_T \frac{\sigma_{xx} + \sigma_{yy}}{2} \, d\Omega,
\]
but now problem may be foreseen. In fact, the average pressure
\[
\int_T p \, d\Omega = \int_T \frac{\sigma_{xx} + \sigma_{yy}}{2} \, d\Omega
\]
is one of the degrees-of-freedom of the JM element and is likely to lead to a locking behavior. In this respect, in fact, there are some similarities with the (locking) \( T_1 P_0 \) element [10] (where displacements are globally \( H^1 \)) and therefore the numerical results to be presented next do not come as a full surprise. Basically, by using the modified JM formulation in the incompressible case, we are requiring at each element level something much similar to the incompressibility condition
\[
\int_T \text{div} \mathbf{u} \, d\Omega = 0,
\]
which is too strong and leads to a locking behavior. The situation is here different since the divergence of the displacement field does not appear explicitly but the presence of constant pressure and linear (globally continuous) displacement field over a triangle is something to be worried about.

4.3. Eigenvalue analysis of a Cook cantilever with the two above formulations

Fig. 4 shows two displacement patterns of a Cook cantilever discretized by \( 16 \times 16 \) elements in the presence of a uniform vertical load. On the left the displacement field one gets with a standard JM formulation is shown, confirming the absence of locking whereas the right picture shows the locking pattern of the same displacement field when global continuity is imposed a priori. To gain insight into the problem, we consider the mixed matrix \( \mathcal{M} \) [10] governing the discrete problem (24), i.e.
\[
\mathcal{M} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix},
\]
where \( A \) and \( B \) are the matrices associated with the bilinear forms \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) previously introduced. As shown in Table 1, there is an easy though illuminating relationship between the number of vanishing eigenvalues of \( \mathcal{M} \) and the mesh size. If \( NZ \) denotes the number of zero eigenvalues of \( \mathcal{M} \) and an \( n \times n \) mesh is considered (amounting to \( 2n^2 \) triangles), one gets
\[
NZ = n \times (n - 1).
\]
By recalling that the left boundary of the Cook cantilever, say $\Gamma_u$, is the only edge where the displacement constraint $u = 0$ is imposed, $n \times (n - 1)$ turns out to be half the number of triangles that do not touch $\Gamma_u$. One gets then the clear indication that each quadrilateral (made up of two adjacent triangles) where no conditions on the displacement field is imposed brings into the problem a null eigenvalue (only when $\lambda$ tends to infinity). The idea is then to let two adjacent triangles share the same pressure value thus leading in fact to a new quadrilateral element to be introduced next.

5. The new element (NV)

5.1. Preliminaries

The above analysis allows one to conclude that every other element pressure is responsible for a locking behavior as it happens in the $T_1P_0$ element [14]. We want therefore to modify the JM element in the cheapest possible way so as to preserve its basic features and gain the capability to handle incompressible materials in the presence of a continuous displacement field. The number of vanishing eigenvalues suggests that every other pressure value on a triangle is “redundant” thus advising to halve the number of independent (averaged) pressure degree of freedoms. The idea is then to imitate the $Q_1P_0$ element with the only slight modification that the bilinear approximation for the displacement on the square is substituted by two distinct and globally continuous linear approximations on adjacent triangles that are glued to form a quadrilateral.

5.2. The NV element

We refer to Fig. 2 where the reference patch for the definition of the NV element is presented. A quadrilateral is formed by gluing two “Johnson–Mercier” triangles by a common edge. Let $T_{ij}$ denote the $j$th subtriangle of the $i$th JM triangle and $Q$ be the global quadrilateral element. We may therefore write

![Fig. 2. Quadrilateral patch for defining the NV element.](image-url)
\[ Q = \bigcup_{i=1}^{2} T_i, \quad T_1 = \bigcup_{j=1}^{3} T_{1j}, \quad T_2 = \bigcup_{j=1}^{3} T_{2j}. \] (33)

The stress space \( \text{NV}(Q) \) is then introduced as

\[
\text{NV}(Q) = \left\{ \tau \mid \tau \in H(\text{div}; Q), \tau|_{T_0} \in [P_1(T_0)]^4, \int_{T_1} p \, d\Omega = \int_{T_2} p \, d\Omega \right\};
\] (34)

where \( p \) is the pressure and one should notice that the condition \( \tau \in H(\text{div}; Q) \) implies that \( \tau \cdot n \) is to be continuous not only across adjacent subtriangles belonging to the same triangle (as in the JM element) but also across the edge shared by \( T_1 \) and \( T_2 \). An element \( e \in \text{NV}(Q) \) is then uniquely determined by the following 25 degrees of freedom [8]:

\[
\int_{e_i} (\tau \cdot n) \cdot p \, ds \quad \forall p \in (P_1(e_i))^2, \quad i = 1, \ldots, 5,
\] (35)

\[
\frac{1}{2} \int_{T_i} \tau_{xx} - \tau_{yy} \, d\Omega, \quad \int_{T_i} \tau_{xy} \, d\Omega, \quad i = 1, 2,
\] (36)

and

\[
\frac{1}{2} \int_{Q} \tau_{xx} + \tau_{yy} \, d\Omega,
\] (37)

where \( e_i \) are the external edges of the quadrilateral and the edge where the two triangles join. We remark now that the crucial difference between our new element and two classic JM elements is the adoption of the pressure degree-of-freedom in Eq. (37) over \( Q \) instead of two different average pressures over \( T_1 \) and \( T_2 \). As shown hereafter we are then able to use continuous displacement approximations whereas with the standard JM element we are not. The displacement field is still linear on each \( T_i \) but now we require global continuity on \( Q \). As to the approximate problem, we define the finite dimensional subspaces of \( H_h \subset H \) and \( V_h \subset V \) by

\[
H_h = \left\{ \tau_h \in H : \tau_h|_\partial \in \text{NV}(Q), \int_{\Omega} \text{tr} \tau_h \, dx = 0 \right\},
\]

\[
V_h = \left\{ v_h : v_h|_{T_i} \in [P_1(T_i)]^2, \quad i = 1, 2, v_h \in H_0^1(\Omega) \right\}.
\]

Therefore, the discrete elastic problem reads: find \((\mathbf{g}_h, \mathbf{u}_h) \in H_h \times V_h\) such that

\[
a(\mathbf{g}_h, \tau_h) + (\mathbf{u}_h, \text{div} \tau_h) = 0 \quad \forall \tau_h \in H_h,
\]

\[
(\text{div} \mathbf{g}_h, \mathbf{v}_h) + (\mathbf{g}_h, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in V_h.
\] (38)

5.3. Details on the implementation

The element may of course be implemented as is on a reference quadrilateral domain. If an implementation of the standard JM element is available, however, it may prove convenient to use an augmented Lagrangian formulation where a constraint of type
\[
\frac{\int_{\Omega_1} \frac{\sigma_{xx} + \sigma_{yy}}{2} \, d\Omega}{\int_{\Omega_1} d\Omega} = \frac{\int_{\Omega_2} \frac{\sigma_{xx} + \sigma_{yy}}{2} \, d\Omega}{\int_{\Omega_2} d\Omega}
\]
is added to the solving system every other triangle of the mesh.

6. Numerical results

6.1. Cook cantilever

The Cook cantilever shown in Fig. 3 is analyzed. It is clamped on the left side, free on the other three sides and acted upon by a uniform vertical load. The Young modulus is \(E = 70\) whereas Poisson ratios \(v = 0.3, 0.4\) and \(0.5\) are considered. At this regard, one should notice that the adopted variational formulation allows for the analysis of incompressible materials with no further adjustments, the only modification being the bilinear form \(a(\cdot, \cdot)\) that reads in the incompressible case

\[
a(\sigma, \tau) = \int_{\Omega} \frac{1}{2\mu} \sigma^{D} : \tau^{D} \, dx.
\]

6.1.1. Computation of displacement and stress fields for given mesh and material properties

The cantilever was uniformly discretized by means of meshes of size \(2^n \times 2^n\), \(n = 2, 3, 4, 5\). Before gaining insight into the convergence analysis, the capabilities of the implemented element are preliminary shown by means of a few results pertaining to the \(16 \times 16\) mesh and \(v = 0.5\). The problem was solved by using several approaches, i.e.

1. classical displacement approach,
2. standard JM formulation,
3. JM stresses and globally \(H^1\) piecewise linear polynomials,
4. NV formulation,
5. B-Barrier method as described and implemented in [19],
6. enhanced-strain formulation as described in [16,19].

Fig. 4 shows two typical locking-free and locking displacement patterns. With reference to the approaches mentioned above, formulations (1) and (3) exhibit a locking behavior whereas the others are locking free. In the absence of an analytical solution to our problem, we are therefore going to use the B-Barrier solution with dense mesh ($128 \times 128$) as a term of comparison to assess the performance of the NV
As to the stresses, Fig. 5 displays the stress flux pattern \( \sigma \cdot n \) computed via NV approach for a \( 16 \times 16 \) mesh. Furthermore, Fig. 8 shows the variation of \( \sigma_{xx} \), \( \sigma_{yy} \). It is in excellent agreement with Fig. 9 that displays the stresses \( \sigma_{xx} \) and \( \sigma_{yy} \) as computed with the B-Bar method.

6.1.2. Convergence analysis of displacement and stress fields with respect to the mesh size for given material properties

To assess displacement convergence rates of NV and JM elements we have considered the upper right corner \( P \), see Fig. 3. For the case \( \nu = 0.5 \), i.e. \( \lambda = \infty \), Fig. 6 shows the convergence of the displacement components \( u_x(P) \) and \( u_y(P) \) for meshes of type \( n \times n \), \( n = 2, 4, 8, 16, 32 \) for both the NV and JM formulations showing basically the same degree of convergence. The displacement error norm versus the mesh size is showcased in the logarithmic plots presented in Fig. 7 where a quadratic convergence rate is found.

![Graph showing convergence of displacement components](image-url)
for the NV element. We may then conclude that, as to the displacement field, the NV and JM elements perform nearly the same way.

As to the stress, though, a few considerations are in order before examining the relevant numerical results. It is well known that a necessary condition for the convergence and stability of any mixed approximation scheme is the ellipticity on the kernel of the bilinear form \( a(\cdot, \cdot) \), see Eq. (19), that holds true in the discrete case whenever the inclusion property

\[
\text{Ker} B_h \subset \text{Ker} B
\]  

is satisfied (of course if ellipticity on the kernel of the continuous problem is guaranteed). One may show that the original element of JM [15] leads to a formulation that satisfies the condition (39). Whether the NV element satisfies (39) or not is still an open question to be tackled and deeply investigated in [8]. The displacement space of the JM element is much larger than the one of the NV element and therefore the

---

**Fig. 8.** NV formulation: \( \sigma_{xx} \) (left) and \( \sigma_{yy} \) (right) distributions.

**Fig. 9.** B-Bar formulation: \( \sigma_{xx} \) and \( \sigma_{yy} \) distributions (\( v = 0.4 \)).
opposite holds true for the kernels of the relevant $B$ operators. The ellipticity of $a(\cdot, \cdot)$ on the kernel of $B$ is therefore harder to be satisfied in the case of the NV element. For the time being, it suffices to say that we have found numerically that using the NV element leads to a slightly lower convergence order for the stress rather than using the original JM formulation. This is basically due to the ellipticity-on-the-kernel condition, see [8]. In fact Figs. 10–12 show the $L^2$ norm error of $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$, $p$ and $\sigma_D$, respectively, for $\nu = 0.333, 0.4$ and 0.5 for both the NV and JM elements. It is then easy to see that when the material is incompressible, i.e. for $\nu = 0.5$ the two elements show the same accuracy whereas the JM element performs slightly better in the case of compressible elasticity. This is confirmed by Tables 2–4 that present the NV and JM $L^2$ norm of the stress error for different meshes in the cases $\nu = 0.333, 0.4$ and 0.5, respectively.
Fig. 12. NV formulation: deviatoric stress norm error vs. mesh size.

### Table 2
$L^2$ norm NV and JM stress error – $v = 0.333$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|\sigma_{nm} - \sigma_{nm}^{NV}|_{L^2}\Omega$</th>
<th>$|\sigma_{nm} - \sigma_{nm}^{JM}|_{L^2}\Omega$</th>
<th>$|\sigma_{nm} - \sigma_{nm}^{JM}|_{L^2}\Omega$</th>
<th>$|p - p^0|_{L^2}\Omega$</th>
<th>$|\sigma - \sigma_D|_{L^2}\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NV</td>
<td>JM</td>
<td>NV</td>
<td>JM</td>
<td>NV</td>
<td>JM</td>
</tr>
<tr>
<td>2</td>
<td>0.8750</td>
<td>0.8649</td>
<td>0.4344</td>
<td>0.5281</td>
<td>0.4915</td>
</tr>
<tr>
<td>4</td>
<td>0.6662</td>
<td>0.4602</td>
<td>0.3109</td>
<td>0.2716</td>
<td>0.3655</td>
</tr>
<tr>
<td>8</td>
<td>0.3896</td>
<td>0.2663</td>
<td>0.2291</td>
<td>0.1885</td>
<td>0.2307</td>
</tr>
<tr>
<td>16</td>
<td>0.2082</td>
<td>0.1552</td>
<td>0.2064</td>
<td>0.1802</td>
<td>0.1468</td>
</tr>
</tbody>
</table>

### Table 3
$L^2$ norm NV and JM stress error – $v = 0.4$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|\sigma_{nm} - \sigma_{nm}^{NV}|_{L^2}\Omega$</th>
<th>$|\sigma_{nm} - \sigma_{nm}^{JM}|_{L^2}\Omega$</th>
<th>$|\sigma_{nm} - \sigma_{nm}^{JM}|_{L^2}\Omega$</th>
<th>$|p - p^0|_{L^2}\Omega$</th>
<th>$|\sigma - \sigma_D|_{L^2}\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NV</td>
<td>JM</td>
<td>NV</td>
<td>JM</td>
<td>NV</td>
<td>JM</td>
</tr>
<tr>
<td>2</td>
<td>0.8617</td>
<td>0.8704</td>
<td>0.4467</td>
<td>0.5387</td>
<td>0.4922</td>
</tr>
<tr>
<td>4</td>
<td>0.6447</td>
<td>0.4690</td>
<td>0.3062</td>
<td>0.2875</td>
<td>0.3650</td>
</tr>
<tr>
<td>8</td>
<td>0.3811</td>
<td>0.2748</td>
<td>0.2142</td>
<td>0.1900</td>
<td>0.2310</td>
</tr>
<tr>
<td>16</td>
<td>0.2106</td>
<td>0.1606</td>
<td>0.1814</td>
<td>0.1557</td>
<td>0.1422</td>
</tr>
</tbody>
</table>

### Table 4
$L^2$ norm NV and JM stress error – $v = 0.5$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|\sigma_{nm} - \sigma_{nm}^{NV}|_{L^2}\Omega$</th>
<th>$|\sigma_{nm} - \sigma_{nm}^{JM}|_{L^2}\Omega$</th>
<th>$|\sigma_{nm} - \sigma_{nm}^{JM}|_{L^2}\Omega$</th>
<th>$|p - p^0|_{L^2}\Omega$</th>
<th>$|\sigma - \sigma_D|_{L^2}\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NV</td>
<td>JM</td>
<td>NV</td>
<td>JM</td>
<td>NV</td>
<td>JM</td>
</tr>
<tr>
<td>2</td>
<td>0.9118</td>
<td>0.9476</td>
<td>0.4964</td>
<td>0.5726</td>
<td>0.4940</td>
</tr>
<tr>
<td>4</td>
<td>0.6068</td>
<td>0.5496</td>
<td>0.3225</td>
<td>0.3859</td>
<td>0.3687</td>
</tr>
<tr>
<td>8</td>
<td>0.3857</td>
<td>0.3638</td>
<td>0.2429</td>
<td>0.2780</td>
<td>0.2427</td>
</tr>
<tr>
<td>16</td>
<td>0.2663</td>
<td>0.2662</td>
<td>0.1986</td>
<td>0.2062</td>
<td>0.1467</td>
</tr>
</tbody>
</table>
6.2. Numerical inf–sup testing

The numerical evaluation of the inf–sup condition has received considerable attention in the recent past [5,10]. Although not equivalent to the analytical proof [10], the numerical test gives however reliable indications on whether the inf–sup condition is fulfilled or not by a given finite-element discretization. Leaving the details of the numerical inf–sup test for the NV element to [18], we hereafter report the main ideas on this issue following [10] and the results of a first successful numerical test. Let \( r \) be the rank of matrix \( B \) in Eq. (31) and consider the standard generalized eigenvalue problems

\[
BT^{-1}B'q_i = \mu_i^2 Sq_i, \quad 1 \leq i \leq r,
\]

\[
B'S^{-1}Bv_i = \mu_i^2 Tv_i, \quad 1 \leq i \leq r,
\]

in which \( S \) and \( T \) are the matrices associated with the scalar product of \( H_h \) and \( V_h \) in Eq. (38). In order for the numerical inf–sup test to be passed “the smallest nonzero eigenvalue (of the eigenvalue problem (40)) must remain bounded away from zero when the dimensions of the spaces increase” [10]. Fig. 13 shows that the lowest eigenvalue remains bounded away from zero when NV elements are used to analyze the Cook cantilever.

Fig. 14. A pinched ring.
6.3. A thin pinched ring

The thin pinched-ring problem is then considered, see Fig. 14 and [6]. The analytical radial deflection under point load is found to be

\[ w = -\frac{PR^3}{8\pi D} (\pi^2 - 8), \quad \text{where} \quad D = \frac{E\ell^3}{12(1 - \nu^2)}, \]

where \( R = 4.953 \) is the mid radius, \( t = 0.094 \) is the thickness, \( E = 10.5 \times 10^6 \) is the Young modulus, \( \nu \) is the Poisson ratio and \( P = 100 \) is the self-equilibrated load. Fig. 15 shows the locking-free displacement pattern for the cases \( \nu = 0.333 \) and \( \nu = 0.499999 \) that allow one to conclude that the element performance is invariant with respect to the Poisson ratio, even in the nearly incompressible and incompressible range.

7. Concluding remarks and future work

Within the framework of locking-free numerical strategies for incompressible media, we have presented the new NV equilibrium finite element as an appealing alternative to more classical existing ones. Moving from the JM element, ad hoc modifications were conceived in order to allow continuous displacement approximations. Not only is this way locking avoided but also the number of displacement degrees-of-freedom is dramatically reduced. Works currently in progress on the themes dealt with in the paper include:

- an extensive numerical inf–sup test of the element [18] along the path of [5];
- a rigorous analysis of the NV element, [8];
- an investigation aimed at the assessment of the element behavior in the large-strain regime [13];
- the development of an adaptive method combining JM and NV elements where the former are used where high displacement and stress gradients appear whereas the latter is adopted in those regions where a “more regular” behavior is found;
- the use of the conventional JM element for modelling fracture phenomena where the inherent discontinuity of the displacement field is computationally appealing.

Acknowledgements

We thank Prof. Franco Brezzi for his guidance during the development of the paper.
References