On the use of stochastic models of uncertainty in active control and structural optimization

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Abstract

This paper deals with applications of random uncertainty models in active control and structural optimization. Uncertainties are modelled via random fields that account for fluctuations of the physical properties of the structure around the mean that is conversely deterministic, i.e. we adopt a pretty conventional approach. Unconventional are conversely the applications proposed. They extend classical results and tools in stochastic optimal control and structural optimization to systems affected by uncertainties. A stochastic-based concept of optimality is set forth and the proposed designs are optimal with respect to this newly established criterion. Numerical studies on MDOF systems and generally orthotropic plates complete the paper validating its theoretical derivations. © 1998 Elsevier Science Ltd. All rights reserved.

1. Introduction

A deep maturity has been reached about the analysis of uncertain structural systems thanks to hundreds of contributions in the last decades. The stochastic finite element method (SFEM) is the milestone on which most of the applications are based and several different versions of it are now available (in Refs. [1–3], among others). There actually exist other philosophies according to which structural uncertainties may be modelled such as the convex approach that will not be dealt with herein [4]. More often than not, however, the aim of the investigation has been limited to a probabilistic structural analysis, i.e. the calculation of a few moments of some quantities of physical interest such as stresses or strains. Other equally interesting applications such as optimal design and computation of active controllers for structures in the presence of uncertainty have conversely received much less attention during such a fertile period. The objective of the present paper is therefore to present a few methodologies for the optimal design and active control synthesis of linear uncertain structural systems. In both cases, an additive decomposition of the elastic moduli and mass density into a deterministic and a fluctuating part is the starting point of the methods, as classically happens in the literature. The original part of the paper is the subsequent one where novel methods of optimization and control are proposed that explicitly account for the uncertainty and provide one with robust solutions. The paper is organized as follows. In Section 2, general issues about generating random structural matrices from physical random fields are reviewed. This is the heart of the stochastic finite element method (SFEM), to be used for active control, and of the stochastic Rayleigh–Ritz method (SRRM), that is adopted as an analysis tool within a structural optimization procedure. Then in Section 3, the paper covers separately active control and structural optimization in the presence of uncertainty. Common features that are shared by these two branches are enlightened such as the need for robustly stable solutions and their connection with the random eigenvalue problem.

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2. Methods of stochastic analysis

2.1. Stochastic fields

We will make use of an additive decomposition of any physical property of interest into a deterministic and a random part. As far as an elastic modulus is concerned, say \( E \), one may write \[ \tilde{E}(x) = \bar{E}[1 + \tilde{g}(x)]; \quad x \in \Omega \] (1)

where \( \Omega \) is the system domain and \( \tilde{g}(x) \) is a (homogeneous zero-mean) random field. The foregoing assumption leads naturally to a decomposition of the stiffness matrix into a deterministic and a stochastic contribution. This holds true for any linear elastic problem regardless the discretization method adopted. More details about the application of the SFEM and the SRRM to get the stochastic stiffness matrix from the introduced random fields will be given in Section 2.2. In the same way, the mass density \( \rho \) may be decomposed as

\[ \tilde{\rho}(x) = \bar{\rho}[1 + \tilde{h}(x)]; \quad x \in \Omega \] (2)

thus obtaining the base for an additive decomposition of the mass matrix. In Section 2.2, attention will be focused on the derivation of the stochastic stiffness matrix having in mind that by means of conceptually identical procedures the stochastic mass matrix may computed as well.

2.2. The stochastic stiffness matrix

2.2.1. Stochastic finite element approach

Using standard finite element notations, the compatibility and linear elastic constitutive equations may be respectively written as

\[ \mathbf{e} = \mathbf{B} \mathbf{u} \quad \text{and} \quad \mathbf{\sigma} = \mathbf{D} \mathbf{e} \]

where \( \mathbf{e} \) is the strain tensor, \( \mathbf{\sigma} \) is the stress tensor, \( \mathbf{B} \) is the compatibility operator and \( \mathbf{D} \) is the elasticity matrix depending upon the peculiar system under investigation. By using Eq. (1), the elasticity matrix \( \mathbf{D} \) may be decomposed in its deterministic and fluctuating parts so that the classic element stiffness matrix

\[ \mathbf{K}_i = \int_{\Omega_i} \mathbf{B}' \mathbf{D} \mathbf{B} \, d\Omega \] (3)

may be re-written as

\[ \hat{\mathbf{K}}_i = \int_{\Omega_i} \mathbf{B}' \mathbf{D} \mathbf{B} \, d\Omega + \int_{\Omega_i} \mathbf{B}' \mathbf{D} \mathbf{B} \, d\Omega = \mathbf{K}_i + \Delta \hat{\mathbf{K}}_i \] (4)

in which the first addend is deterministic and the second is stochastic. Once the element stiffness matrix is formed, the global stiffness matrix has to be assembled. Needless to say that this operation is a delicate one since the mesh that governs the deterministic problem need not be same as the one with which random fields are discretized into random variables.

2.2.2. Stochastic Rayleigh–Ritz approach

Even though the method is general in principle, we will present it along with its stochastic extension with explicit reference to laminated composite plates that will be object of optimization later on \[6\]. The fourth order stiffness operator of thin multilayered generally orthotropic plates, say \( \mathcal{S} \) is written in Cartesian coordinates as

\[ \mathcal{S}[w] = D_{11} \frac{\partial^2 w}{\partial x^2} + 2D_{12} \frac{\partial^2 w}{\partial x \partial y} + 2(D_{12} + D_{66}) \frac{\partial^2 w}{\partial y^2} + 4D_{26} \frac{\partial^2 w}{\partial x \partial y^2} + D_{22} \frac{\partial^2 w}{\partial y^4} \] (5)

where \( w \) is the midplane deflection and the coefficients \( D_{ij} \) may be evaluated following the classical lamination theory. They depend on four elastic moduli, i.e. \( E_{11}, E_{22}, \nu_{12} \) and \( G \) that are going to be modelled as random fields herein. Firstly an expression for the stiffness matrix is to be derived in the deterministic case. To this goal a set of admissible functions, satisfying the geometric boundary conditions is introduced. If \( N \) denotes the number of Ritz functions, one has

\[ \mathcal{S} = \{ \mathbf{Z}_k(x, y), \; k = , \ldots , N \} \] (6)

so as to expand the candidate solution as a Ritz sequence, i.e.

\[ w(x, y) = \sum_{k=1}^{N} a_k \mathbf{Z}_k(x, y) \] (7)

By standard Gauss–Green formulae, the stiffness matrix associated with \( \mathcal{S} \) after Rayleigh–Ritz discretization may be written as

\[ K_{ij} = \int_{\Omega} \left[ D_{11} \mathbf{Z}_{i,x} \mathbf{Z}_{j,x} + D_{12} (\mathbf{Z}_{i,x} \mathbf{Z}_{j,y} + \mathbf{Z}_{i,y} \mathbf{Z}_{j,x}) ight. \\
+ 4D_{26} \mathbf{Z}_{i,y} \mathbf{Z}_{j,x} + 2D_{66} (\mathbf{Z}_{i,x} \mathbf{Z}_{j,y} + \mathbf{Z}_{i,y} \mathbf{Z}_{j,x}) \\
+ \left. 2D_{22} (\mathbf{Z}_{i,y} \mathbf{Z}_{j,y} + \mathbf{Z}_{i,x} \mathbf{Z}_{j,x}) + D_{22} \mathbf{Z}_{i,x} \mathbf{Z}_{j,x} \right] d\Omega \] (8)

or, after introducing the single index notation,

\[ D_{11} \mathbf{D}_{11}, \quad D_{22} \mathbf{D}_{22}, \quad D_{12} \mathbf{D}_{12}, \quad D_{66} \mathbf{D}_{66}, \quad D_{26} \mathbf{D}_{26}, \quad D_{22} \mathbf{D}_{22} \] (9)

in compact form as

\[ K_{ij} = \sum_{k=1}^{N} \int_{\Omega} \mathbf{D} \mathbf{D} \mathbf{Z}_k(x, y) d\Omega = \int_{\Omega} \mathbf{D} \mathbf{D} \mathbf{Z}_k(x, y) d\Omega \] (10)

where the vector \( \mathbf{D} \mathbf{Z}_k \) entails products of second order derivatives of Ritz functions and may be written as
Hereafter the four elastic fields are given a random field description of the type

\[ E_i = E_{i0} [1 + f_1(x, y)], \]
\[ E_2 = E_{20} [1 + f_2(x, y)], \]
\[ \nu_{12} = \nu_{120} [1 + f_3(x, y)], \]
\[ G_{i2} = G_{i20} [1 + f_4(x, y)] \]

(11)

where a subscript "0" denotes the reference deterministic value and the \( f_i \)’s are zero-mean homogeneous random fields. Finding the link between Eq. (8) and Eq. (11) is not so straightforward. The problem is that in Eq. (8) the elastic moduli are hidden into the expressions of the coefficients \( D_i \). Therefore, one should operate on the expression of the \( D_i \) first to get their random extension and then average through the thickness to get the stochastic stiffness matrix. To this goal, let us consider standard results from classical lamination theory that allow one to write

\[ D_i = \frac{1}{3} \sum_{k=1}^{NQ} (\tilde{Q}_i)_k (z_0^k - z_{k-1}^k); \quad i = 1, \ldots, 6 \]

(12)

\[ \tilde{Q}_i = \sum_{k=1}^{NQ} g_{ik}(\theta) Q_k; \quad i = 1, \ldots, 6 \]

(13)

where \( NQ = 4 \) is the number of \( Q \), or \( g_{ik}(\theta) \) depend on the lamination angle \( \theta \) and

\[ Q_1 = \frac{E_1}{1 - \nu_{12} E_2}; \quad Q_2 = \frac{\nu_{12} E_1}{1 - \nu_{12} E_2}; \]
\[ Q_3 = \frac{E_2}{1 - \nu_{12} E_2}; \quad Q_4 = G \]

Since \( Q_i \) depends nonlinearily on the elastic moduli \( E_1, E_2, \nu_{12} \) and \( G_{12} \), a linearization about the mean values of the elastic moduli is performed for simplicity sake:

\[ Q_k = Q_{k0} + \sum_{r=1}^{NV} Q_{kr} | f_r \]

(14)

where the comma indicates partial derivative, | denotes evaluation of a random quantity for the mean values of the relevant stochastic fields and \( NV = 4 \) is the number of stochastic fields entering the formulation. If Eq. (14) is now substituted into Eq. (13), one gets

\[ \tilde{Q}_i = \sum_{k=1}^{NQ} g_{ik}(\theta) Q_k = \sum_{k=1}^{NQ} \left[ g_{ik}(\theta) \left( Q_{k0} + \sum_{r=1}^{NV} Q_{kr} | f_r \right) \right] \]
\[ = \sum_{k=1}^{NQ} g_{ik}(\theta) Q_{k0} + \sum_{k=1}^{NQ} \sum_{r=1}^{NV} g_{ik}(\theta) Q_{kr} | f_r \]
\[ = \tilde{Q}_i^{\text{det}} + \tilde{Q}_i^{\text{stoc}}; \quad i = 1, \ldots, 6 \]

(15)

Thanks to Eq. (12), one finally obtains

\[ D_i = \frac{1}{3} \sum_{n=1}^{NL} (\tilde{D}_i)_n (z_n^3 - z_{n-1}^3) \]
\[ = \frac{1}{3} \sum_{n=1}^{NL} (z_n^3 - z_{n-1}^3) \left[ \sum_{k=1}^{NQ} g_{ik}(\theta) Q_{k0} \right] \]
\[ + \frac{1}{3} \sum_{n=1}^{NL} (z_n^3 - z_{n-1}^3) \left[ \sum_{k=1}^{NQ} \sum_{r=1}^{NV} g_{ik}(\theta) Q_{kr} | f_r \right] \]
\[ = D_i^{\text{det}} + D_i^{\text{stoc}}; \quad i = 1, \ldots, 6 \]

(16)

The stochastic part of the stiffness matrix may therefore be written as

\[ K_{ij}^{\text{stoc}} = \frac{1}{3} \sum_{n=1}^{NL} (z_n^3 - z_{n-1}^3) \left[ \sum_{r=1}^{NV} \sum_{k=1}^{NQ} g_{ik}(\theta) \left( \Psi_{ij} Q_{kr} | f_r \right) d\Omega \right] \]

(17)

If the reference values of the stochastic fields are uniform in \( \Omega \), the quantities \( Q_{kr} \) are constant and therefore a set of coefficients may be defined as

\[ \tilde{Q}_{kr} = \frac{1}{3} \sum_{n=1}^{NL} (z_n^3 - z_{n-1}^3) \left[ \sum_{k=1}^{NQ} g_{ik}(\theta) Q_{k0} \right] \]

(18)

that allows us to write

\[ K_{ij}^{\text{stoc}} = \sum_{h=1}^{NV} \sum_{r=1}^{NV} \tilde{Q}_{hr} \int_{\Omega} \Psi_{ij} f_r d\Omega \]

(19)

3. Linear quadratic Gaussian (LQG) control for uncertain systems

LQG theory is based on the solution of the quadratic programming differential problem

\[ \min J = \int_{t_0}^{t_f} \left[ \frac{1}{2} \dot{X}'QX + \mathbf{u}'R\mathbf{u} \right] dt \]

(20)

s.t. \( \dot{X} = A X + B u + C_g \)

(21)

where \( X \) is the state vector of the system, \( u \) the control vector, \( A \) the structural matrix, \( C_g \) the disturbance
vector and \( \mathbf{B} \) a topological matrix providing the effect of the control action \( \mathbf{u} \) on each degree-of-freedom. Moreover, \( \mathbf{Q} \) and \( \mathbf{R} \) are weighting matrices on which the closed loop poles of the system depend and \((t_0, t)\) is the time interval in which the earthquake episode occurs. The structural matrix, the state vector and the excitation vector may be respectively written as

\[
\mathbf{A} = \begin{bmatrix}
-o_o^2 & 1 & 0 & 0 \\
-o_o^2 & 0 & 0 & 0 \\
\mathbf{0} \times \mathbf{2} & 2 \sigma_x^2 \sigma_y^2 & 0 & 0 \\
\mathbf{0} \times \mathbf{2} & 2 \sigma_y^2 \sigma_z^2 & 0 & 0 \\
\end{bmatrix}
\]

(22)

\[
\mathbf{X} = \begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
x \\
x \\
\end{bmatrix}, \quad \mathbf{C}_x = \begin{bmatrix}
o & \mathbf{0} \\
\mathbf{0} & \mathbf{1} \\
\end{bmatrix}
\]

(23)

where \( \mathbf{r} \) is a vector which distributes the base excitation to the degrees-of-freedom of the system. The minimization of the functional \( J \) under the constraint of the equation of motion involves a Riccati equation to be written next. This is still true when the excitation is a nonstationary response of the system.

The sensitivities of the closed loop system matrix with respect to the random vector \( \mathbf{e} \), may thus be computed as

\[
\mathbf{A}_0 = \mathbf{A}|_{\mathbf{e}=0}, \quad \mathbf{A}_i = \frac{\partial \mathbf{A}}{\partial \mathbf{e}}|_{\mathbf{e}=0}, \quad \mathbf{A}_{i,j} = \frac{\partial^2 \mathbf{A}}{\partial \mathbf{e}_i \partial \mathbf{e}_j}|_{\mathbf{e}=0}
\]

(32)

Notice that \( \mathbf{A}_0 = \mathbf{A}|_{\mathbf{e}=0} \) is the deterministic part of the stochastic system matrix. When computing the optimal gain matrix, the structural parameters are assigned and the functional to be minimized is

\[
J = E \left[ \int_{t_0}^{t} \left[ \dot{\mathbf{X}}^T \mathbf{Q} \dot{\mathbf{X}} + \mathbf{u}^T \mathbf{R} \mathbf{u} \right] dt \right]
\]

(33)

An approximate solution of the problem is obtained putting \( \mathbf{e} = 0 \), i.e. solving the Riccati matrix equation with \( \mathbf{A}_0 \) as system matrix,

\[
\mathbf{P}_0 = \mathbf{A}_0^T \mathbf{P}_0 + \mathbf{P}_0 \mathbf{A}_0 + \mathbf{H}^T \mathbf{R} \mathbf{H} - \mathbf{Q}
\]

(34)

3.2. Exact solution for the conditional response

If one introduces the closed loop matrix \( \mathbf{A}^* = \mathbf{A} - \mathbf{B} \mathbf{x} \), the equation of motion may be re-written as

\[
\ddot{\mathbf{X}}(\mathbf{e}) = \mathbf{A}^* \mathbf{X}(\mathbf{e}) + \mathbf{B} \mathbf{u}(\mathbf{e}) + \mathbf{C}_x = \mathbf{A}^* \mathbf{X}(\mathbf{e}) + \mathbf{C}_x
\]

(35)

The sensitivities of the closed loop system matrix with respect to the random vector \( \mathbf{e} \), may thus be computed as

\[
[A^*]_0 = \mathbf{A}|_{\mathbf{e}=0} - \mathbf{B} \mathbf{K}, \quad [A^*]_i = \frac{\partial \mathbf{A}}{\partial \mathbf{e}_i}|_{\mathbf{e}=0}, \quad [A^*]_{i,j} = \frac{\partial^2 \mathbf{A}}{\partial \mathbf{e}_i \partial \mathbf{e}_j}|_{\mathbf{e}=0}
\]

(36)

The covariance matrix \( \Sigma \) of such system conditional on
whereas mean and variance of obtained as total variance of a response quantity, say 3.3. Total response statistics

stationary Lyapunov matrix equation as a step-by-step function. The nonstationary Lyapunov equations above may be conveniently solved via a backward difference scheme as

\[
\frac{\partial \sigma}{\partial t} = A \sigma + \sigma A' + C
\]

that allows the computation of the solution of a nonstationary Lyapunov matrix equation as a step-by-step evolution of stationary solutions.

3.3. Total response statistics

Given the conditional second statistics above, the total variance of a response quantity, say \( \hat{R} \), may be obtained as

\[
Var[\hat{R}] = E[Var[\hat{R}|\varepsilon = \varepsilon]] + Var[E[\hat{R}|\varepsilon = \varepsilon]]
\]

where \( E[\hat{R}|\varepsilon = \varepsilon] \) and \( Var[\hat{R}|\varepsilon = \varepsilon] \) are the conditional mean and variance of \( R \) given \( \varepsilon = \varepsilon \), respectively, whereas \( E_\varepsilon \) and \( var_\varepsilon \) are the expectation and the variance with respect to \( \varepsilon \). The latter equation, coupled with the second order approximation herein adopted, allows to calculate the total variance of the response as

\[
Var[\hat{R}] \approx Var[\hat{R}]_{\varepsilon=0} + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 Var[\hat{R}]}{\partial \varepsilon_i \partial \varepsilon_j} E[\varepsilon_i \varepsilon_j]
\]

3.4. Numerical example

The vibrations of a cantilever of length \( L = 4 \) m are examined under the action of a base ground motion. Standard beam elements are used to discretize the system and obtain the global stiffness and mass matrices. The latter is a consistent one, i.e. rotational inertial dofs are included although no rotational component of the excitation was considered. The mean of the mass density is \( \bar{\rho} = 10,000 \) kg/m and that of the Young modulus is \( E = 2.1 \times 10^6 \) KN/m². The coefficient of variation of the Young modulus and the mass density is 0.3. A single actuator located at the bottom end of the beam was considered. The earthquake ground motion is modelled as a Gaussian white noise power spectral density function. The nonstationary Lyapunov equations above may be conveniently solved via a backward difference scheme as

\[
\frac{\partial \sigma}{\partial t} = A \sigma + \sigma A' + C
\]

4. Optimal robust design of laminated paltes wrt eigenvalues

If \( \lambda \) denotes the set of eigenvalues of the operator \( A \), the angles governing ply orientations and \( h \) layer thicknesses, then a general deterministic optimal design problem may be set forth as

\[
\min_{\lambda, h} f(\lambda(\theta, h))
\]

such that \( -\pi \leq \theta_i \leq \pi \) \( i = 1, \ldots, N \)

and \( h_{\min} \leq h_i \leq h_{\max} \) \( i = 1, \ldots, N \)

of which possible particularizations are

\[
(1) f(\lambda) = \max[\min \lambda]
\]

\[
(2) f(\lambda) = \max[\lambda_i - \lambda_j], \quad i, j \text{ fixed}
\]

\[
(3) f(\lambda) = \lambda_i + \pi \left( m \sum_{i=1}^{m} \frac{\omega_y}{\omega_i - \omega_j} \right)^{-1}
\]

(1) Classical max–min problem for Euler load or fundamental frequency; (2) spectral separation
Fig. 1. Uncontrolled and controlled free-end cantilever variance.

Fig. 2. Evolution of displacement variance with time along the cantilever. (Uncontrolled = solid, controlled = dashed).
4.1. Need for robust optimal solutions

Eigenvalue optimization problems are known to be extremely sensitive with respect to slight deviations from nominal values. The sensitivity of such problem is tremendous and the scenario is made even more complicated by the loss of regularity and smoothness that takes place in the case of eigenvalue crossing in

requisite to avoid mode interaction and (3) regular objective to gain differentiability.

In what follows, motivations that call for a robustified version of the problem above are given along with computationally viable strategies for its extension and solution. Hereafter robust optimal denotes an optimality condition that takes into explicit account the uncertainty in the structural model.

Fig. 3. Space–time controlled displacement variance (without uncertainties).

Fig. 4. Space–time controlled displacement variance (only uncertainties).
the design space. Therefore, the standard assumption of neglecting the uncertainties in material parameters may lead to extremely poor designs. The objective is to propose a new methodology capable of handling material uncertainty explicitly, i.e. providing the designer with a robust optimal solution. The key idea is to endow the design space, i.e. $x = \{\theta, h\}$, with a probability-based measure of sensitivity with respect to material properties randomness. If $\mu(\theta, h)$ denotes such a measure, one may then set up and solve a robustified version of the optimal problem in which material uncertainty is explicitly accounted for. Once $\mu(\theta, h)$ is

Fig. 5. Space–time controlled rotation variance (without uncertainties).

Fig. 6. Space–time controlled rotation variance (only uncertainties).
computed, refined robust optimization problems may be written as:

**Additive approach:**

\[
\min_{\theta, h} [\lambda(\theta, h)] + (1 - \alpha)g(\mu(\theta, h))
\]

such that 
- \( -\pi \leq \theta_i \leq \pi \) for \( i = 1, \ldots, N \)
- \( h_{\min} \leq h_i \leq h_{\max} \) for \( i = 1, \ldots, N \)

**Multiplicative approach:**

\[
\min_{\theta, h} [\alpha(\theta, h)]g(\mu(\theta, h))
\]

such that 
- \( -\pi \leq \theta_i \leq \pi \) for \( i = 1, \ldots, N \)
- \( h_{\min} \leq h_i \leq h_{\max} \) for \( i = 1, \ldots, N \)

**Augmented design space approach:**

\[
\min_{\theta, h, \xi} [\lambda(\theta, h)] + (1 - \alpha)g(\xi)
\]

such that 
- \( -\pi \leq \theta_i \leq \pi \) for \( i = 1, \ldots, N \)
- \( h_{\min} \leq h_i \leq h_{\max} \) for \( i = 1, \ldots, N \)

where \( \alpha \) is the convex combination parameter that weights the robustness of the formulation with respect to the original objective and \( g \) is a scalar-valued function selected on the basis of the results of the stochastic analysis.

### 4.2. Choosing the coefficient of variation (COV) as \( \mu \)

For a complete derivation of this topic we refer to Ref. [9]. Without entering details, it suffices here to say that a numerical procedure is available for computing the coefficient of variation of any objective function with respect to material properties randomness. The coefficient of variation is then used as the simplest measure of sensitivity to material randomness as shown in the following numerical study.

### 4.3. Numerical example

We consider a single lamina of given thickness, the design variable being ply orientation. The objective of the design is the maximization of the fundamental frequency of vibration. The nominal properties of the lamina are \( E_1 = 2.06 \times 10^5 \) N/mm\(^2\), \( E_2 = 2.06 \times 10^4 \) N/mm\(^2\), \( v_{12} = 0.3 \) and \( G_{12} = 5.15 \times 10^3 \) N/mm\(^2\). A square lamina is considered first. Fig. 7 presents the variation of the fundamental frequency with respect to the angle of lamination. The optimal solution is shown to be \( \theta = 0 \) and \( \theta = 90^\circ \). This is to be contrasted with Fig. 8 where the coefficient of variation of the fundamental frequency is showed. The optimal solution happens to be the less sensitive to scatter in material properties and therefore the classical optimal solution is also robustly optimal.

![Fig. 7. Fundamental frequency of the square lamina.](image-url)
Fig. 8. Coefficient of variation of the fundamental frequency of the square lamina.

Fig. 9. Fundamental frequency of the rectangular lamina.
Fig. 10. Coefficient of variation of the fundamental frequency of the rectangular lamina.

Fig. 11. Modified objective function $f_{0.2}(\theta)$. 
Fig. 12. Modified objective function $f_{0.4}(\theta)$.

Fig. 13. Modified objective function $f_{0.6}(\theta)$.
A similar analysis is then worked out for a rectangular plate with aspect ratio equal to 5. Figs. 9 and 10 lend themselves to a completely different interpretation. The optimal solution, i.e. $\pi/2$, is also the most sensitive and therefore the classical optimal solution is not a robust one.

To find a solution that is classically and robustly optimal at the same time, a new objective function is introduced as

$$f_\alpha(\theta) = \sqrt{\lambda_1(\theta)} e^{-\alpha \text{COV}(\theta)}$$  \hspace{1cm} (52)

where $\alpha$ is a scalar parameter that weights classical optimality with respect to robust optimality. Figs. 11–13 clearly show that the robust optimal solution is no longer $\theta = \pi/2$ but has moved backward.

5. Concluding remarks

Classic issues in structural control and optimization have been herein extended to include explicit treatment of material uncertainty. To this goal, stochastic additive models of uncertainty were used and coupled to original extensions of well-established methods of active control (LQG) and structural optimization (optimal design with respect to eigenvalues). It was shown that the notion of classical optimality does not always coincide with the concept of robust optimality. As to the active control, the effect of structural uncertainties on controlled systems was assessed by means of second order analyses on a continuous cantilever. Numerical evidence has demonstrated that the sensitivity of the closed-loop structure is tremendously decreased when contrasted to the uncontrolled one.

References