A Wavelet–Galerkin Method for Elastoplasticity Problems

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Abstract. We use biorthogonal B-spline wavelet bases to discretize the dynamic response problem for a straight elasto-plastic rod. In an elastic predictor/plastic corrector method, we use interpolatory wavelets for the stress correction. In several numerical experiments, we show the potential of Wavelet–Galerkin methods for elastoplasticity problems.

1 Introduction

The numerical treatment of elastoplasticity problems is still a very challenging field of active research. Besides classical methods such as Finite Differences, Finite Elements, Spectral Elements and others, quite recently first attempts using wavelet bases have been made, [11, 21].

In recent years, significant progress has been made for using Wavelet–Galerkin methods for the numerical solution of certain operator equations including elliptic partial differential equations, boundary integral equations and also saddle point problems. In fact, wavelets have been proven to give rise to a diagonal multilevel preconditioner for elliptic operators such that the condition number of the preconditioned operators is asymptotically optimal [15, 17, 23]. More recently, adaptive wavelet strategies for elliptic and saddle point operators were introduced that have proven to converge and to guarantee asymptotically optimal efficiency [8, 12, 13]. These adaptive schemes have been successfully implemented and tested for 1d and 2d problems, [4].

One key ingredient for the analysis is the use of biorthogonal wavelet bases within a Galerkin discretization. That is why we decided not to use a collocation method based on interpolatory wavelets as considered in [6] for elliptic problems and in [24] for structural mechanics problems. Here, we use an elastic predictor/plastic corrector method in terms of a stress correction. This correction has to be done pointwise. Hence, we use interpolatory wavelets in the correction step and perform fast change of bases to switch between the different representations.

We first review in Section 2 the governing equations. Then, we detail the discretization methods both in time and space and recall the basic facts on wavelet bases in Section 3. Finally, in Section 4, we perform several numerical tests. In particular we compare the use of wavelet bases of different orders (also including high order bases). Moreover, we consider the sizes of the wavelet coefficients of the solution. The results indicate a great potential for using adaptive wavelet methods which will be considered in a forthcoming paper. Let us finally mention that we cannot expect a similar analytical background for elastoplasticity problems which is available for elliptic operator equations since the mathematical theory for the latter equations is by far more explored.
2 THE PHYSICAL PROBLEM

We study the dynamic response of a straight elastoplastic rod. In this section, we briefly review the governing equations.

Referring to Figure 1, the space variable is denoted by \( x \), the time variable by \( t \) and \( u(x, t) \) is the axial displacement of the rod. The physical problem is governed by classical relations expressing equilibrium between applied forces and induced internal stresses, compatibility between displacements and strains and the nonlinear constitutive law that relates stresses to strains. As to equilibrium one may write

\[
(A\sigma)' + f = \rho \ddot{u},
\]

where \( A(x) \) and \( \rho(x) \) are the cross section and the mass density per unit length, \( f(x, t) \) is the axial force and \( \sigma \) is the axial stress. Furthermore, space and time differentiation are indicated by a superposed prime and dot, respectively. The hypothesis of small displacement gradients will be made under which the compatibility condition may be written as

\[ \varepsilon = u', \]

in which \( \varepsilon(x, t) \) is the axial strain of the rod. The elastoplastic constitutive law classically needs to be introduced in incremental form since the stress does not only depend on the current strain as it happens in the purely elastic case, but also on the entire past history of it. For clarity sake, we first remark that the purely linear-elastic problem is governed by a constitutive law that reads

\[ \sigma = E\varepsilon, \]

in which \( E \) is the Young modulus. Therefore, by eliminating stress and strain in (1), (2) and (3), one ends up with a wave equation having the displacement \( u \) as unknown, i.e.

\[
(EA\varepsilon)' + f = \rho \ddot{u}.
\]

We now introduce the basic incremental relationships defining the elastoplastic behavior of the rod. The main hypothesis, widely used and accepted in the literature, \([25, 26]\), is the additive decomposition of the total strain rate \( \dot{\varepsilon} \) into its elastic and plastic contributions, i.e.,

\[ \dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p. \]

The stress rate \( \dot{\sigma} \) may then be written in terms of any of its above described contributions by introducing the tangent modulus \( E_t \) and the plastic one \( E_p \). These are defined by the following relations

\[ \dot{\sigma} = E_t \dot{\varepsilon} = E_t \dot{\varepsilon}^e = E_p \dot{\varepsilon}^p, \]

where Figure 2 visualizes the introduced quantities. Notice that in Figure 2, \( \sigma_{Yt} \) is the so called (tension) yielding stress, i.e., the stress above which the material is no longer elastic but undergoes permanent, irreversible deformations. Furthermore, \( \sigma_{Yt} \) is a variable itself and is to be updated at each time instant according to the so called hardening rule that will be discussed next. From a computational point of view, the difficulty is that one does not know in advance whether a stress or strain increment will cause plastic loading or elastic unloading. We assume
Figure 2: Stress and strain increments

the rod to be stress–free for \( t = 0 \) and to behave elastically as long as \((t, x) \in I\) where \( I \) is the instantaneous elastic domain defined as

\[
I := \{(t, x) \in [0, T] \times [0, 1] : -\sigma_{Y_2}(t, x) \leq \sigma(t, x) \leq \sigma_{Y_1}(t, x)\} \quad (7)
\]

In (7), \( \sigma_{Y_1} \) and \( \sigma_{Y_2} \) are the yielding stresses in tension and compression, respectively, which we group into the vector \( \sigma_Y = [\sigma_{Y_1} \quad \sigma_{Y_2}] \). For \( t = 0 \) the yielding stresses are known from experimental tests and they evolve with time following some hardening rule. The key point is that \( \sigma_{Y_1} \) and \( \sigma_{Y_2} \) depend on each other so that it is convenient to separate the total plastic strain into its tension and compression components as

\[
\varepsilon^p(t, x) = \kappa_1(t, x) - \kappa_2(t, x) = [1 - 1] \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} = : N\kappa(t, x), \quad (8)
\]

where \( N = [1 - 1] \) and \( \kappa_1 \) and \( \kappa_2 \) are non-negative and non-decreasing. One may then write

\[
\dot{\kappa}_i(t, x) \geq 0, \quad \text{for all} \quad (t, x) \in [0, T] \times [0, 1]. \quad (9)
\]

The elastic domain in (7) is redefined next by introducing the so-called plasticity functions \( \varphi = (\varphi_1, \varphi_2) \) which fulfill the relation

\[
\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} := \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sigma - \begin{bmatrix} \sigma_{Y_1} \\ -\sigma_{Y_2} \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

or, in vector form

\[
\varphi(t, x) = N^t \sigma(t, x) - \sigma_Y(t, x) \leq 0, \quad (10)
\]

where the inequality is meant componentwise. Plastic strains may accumulate if and only if both the current stress state and its update belong to the boundary of the elastic domain, i.e., if and only if \( \varphi = 0 \) and \( \dot{\varphi} = 0 \). If this is not the case an elastic phase takes place. Therefore, the following complementarity rules govern the entire process

\[
\varphi_i \leq 0, \quad \dot{\kappa}_i \geq 0, \quad \varphi_i \kappa_i = \dot{\varphi}_i \dot{\kappa}_i = 0. \quad (11)
\]

By differentiating (10), the time rate of the elastic domain may be computed as

\[
\dot{\varphi}_i = \pm (\dot{\sigma} - \dot{\sigma}_{Y_1}) = \pm \left( \dot{\sigma} - \sum_{j=1}^{2} \frac{\partial \sigma_{Y_1}}{\partial \kappa_j} \dot{\kappa}_j \right) = \pm \dot{\sigma} - \sum_{j=1}^{2} \dot{H}_{ij} \dot{\kappa}_j, \quad (12)
\]

or

\[
\dot{\varphi} = N^t \dot{\sigma} - H \dot{\kappa}, \quad (13)
\]
where $H$ is the so-called hardening matrix. In the case of linear hardening, i.e., when the plasticity functions may be written as

$$\varphi(t, x) = N^\prime \sigma(t, x) - H\kappa(t, x) - \sigma_Y(0, x), \quad (14)$$

one gets the following generalized hardening rule

$$\begin{bmatrix} \sigma_{Y_1} \\ \sigma_{Y_2} \end{bmatrix} = \begin{bmatrix} \sigma_{Y_0} \\ \sigma_{Y_0} \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix},$$

or, in vector form,

$$\sigma_Y(t, x) = \sigma_Y(0, x) + H\kappa(t, x). \quad (15)$$

The frequently used case of isotropic and kinematic hardening are recovered as particular cases by choosing $H$ respectively equal to

$$H_{iso} = h \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad H_{kin} = h \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \quad (16)$$

The elastoplastic incremental constitutive law may finally be written as

$$\dot{\sigma} = E(\dot{\varepsilon} - \ddot{\varepsilon}), \quad \ddot{\varepsilon} = N\dot{\kappa},$$

$$\varphi = N^\prime \sigma - \sigma_Y(\kappa), \quad \dot{\varphi} = N^\prime \dot{\sigma} - H\dot{\kappa},$$

$$\varphi \leq 0, \quad \kappa \geq 0, \quad \dot{\varphi} \kappa = 0, \quad \ddot{\varphi} \kappa = 0.$$

Then we obtain our problem under consideration as

**Problem 1** For given $\rho$, $E$, $E_z$, $E_p$, $A$ and $f$ we seek for $u$, $\varepsilon$ and $\sigma$ such that:

$$\rho(x)\ddot{u}(t, x) - (E(x)A(x)u'(t, x))' = f(t, x), \quad (t, x) \in I, \quad (E)$$

and

$$\rho(x)\ddot{u}(t, x) - (A(x)\sigma(t, x))' = f(t, x), \quad (t, x) \notin I, \quad (P)$$

$$\dot{u}'(t, x) - \varepsilon(t, x) = 0, \quad (t, x) \notin I, \quad (P)$$

$$\dot{\sigma}(t, x) - E_z(\sigma(t, x))\dot{\varepsilon}(t, x) = 0, \quad (t, x) \notin I, \quad (P)$$

Finally, we pose the following initial and boundary conditions

$$u(0, x) = u_0(x), \quad u'(0, x) = u_1(x), \quad x \in [0, 1], \quad (B_1)$$

for some given functions $u_0$ and $u_1$, as well as

$$u(t, 0) = u(t, 1) = 0, \quad t \in [0, T], \quad (B_2)$$

3 **Numerical Treatment**

In this section, we are going to describe the numerical treatment of Problem 1. We will detail the elastic predictor–plastic corrector method and we introduce the discretization both in space and time.

### 3.1 Elastic prediction and plastic correction

Let us start by the elastic predictor–plastic corrector strategy. There exist several cases to be handled numerically, all of which are particular cases of the complementarity rule (11). For clarity sake we hereafter focus on one of them, i.e. the case of plastic loading or elastic unloading in tension. Let then $\varphi(t, x) = 0$ for some $t \in [0, T]$ and $x \in [0, 1]$, meaning that the converged stress $\sigma(t, x)$ is on the boundary of the instantaneous elastic domain. Given is also $u(t, x)$ for some time $t$. Then, for a given $\Delta t > 0$,
we compute the elastic predictor \( u^* (t + \Delta t, x) \) by solving the problem (\( E \)) (since we always consider the initial and boundary conditions, we will omit referring to them all the time). Then, by using the second equation in (\( P \)) (see also (2)), we compute \( \varepsilon^* \) and then, by using (3), we obtain the elastic trial stress \( \sigma^* \). If \( \sigma^* < \sigma_Y \) then an elastic unloading has taken place and therefore \( \sigma(t + \Delta t, x) = \sigma^* \) and no correction is required. If conversely \( \sigma^* > \sigma_Y \), a strain-driven modified Newton–Raphson correction scheme is used and illustrated in Figure 3. The stress \( \sigma^* = \sigma(t, x) + E(x)[\varepsilon^*(t, x) - \varepsilon(t, x)] \) is no longer updated and will therefore coincide with the value \( \sigma(t + \Delta t, x) \). A lack of consistency may however be noticed between the so computed stress \( \sigma^* \) and the one which is compatible with the actual stress–strain curve, i.e., \( \sigma^*(t, x) = \sigma(t, x) + E_i(t)\varepsilon^*(t, x) \). The quantity \( \Lambda(x)[\sigma^* - \sigma^*](t, x) \) becomes a virtual, unequilibrated force that is brought to the right-hand side of Equation (\( E \)) so as to allow the computation of a further update for the displacement and for the strain \( \varepsilon \). The procedure ends when the actual stress–strain curve joins the plateau \( \sigma^* \) where the solution in terms of stress, strain and displacement is attained (point \( P(t + \Delta t, x) \) in Figure 3). Notice that \( \sigma^* \) is not only the stress \( \sigma(t + \Delta t, x) \) but also the new value of the yielding stress \( \sigma_Y \) to be used for the subsequent stress computation.

When linear isotropic hardening is adopted, see (16), the whole stress correction procedure is governed by eight alternative cases. We will hereafter describe the tension cases in detail. The remaining (compression) cases can easily be obtained by symmetry arguments and by replacing \( \sigma_Y \) by \( \sigma_Y \). We will be using discrete-time notations and denote by \( \sigma_n \) the computed stress value at time \( t_n \) by \( \sigma^*_{n+1} \) the trial stress value at time \( t_{n+1} \) and by \( \sigma_Y, \sigma_Y \) the current values of the yielding stress. The value of the corrected stress is denoted by \( \sigma^* \).

1. \( 0 \leq \sigma_n \leq \sigma^*_{n+1} \leq \sigma_Y \).
   If \( \sigma_n \) and the trial stress \( \sigma^*_{n+1} \) are both below the yielding stress \( \sigma_Y \), we are still in the elastic range, i.e., no correction has to be performed. This implies \( \sigma^* := \sigma^*_{n+1} \).

2. \( 0 < \sigma_n \leq \sigma^*_{n+1}, \sigma_n < \sigma_Y, \sigma_Y < \sigma^*_{n+1} \).
   In the case that \( \sigma_n \) is below \( \sigma_Y \), but the elastic trial \( \sigma^*_{n+1} \) is above, then there is a transition from the elastic to the plastic regime and the following correction has to be made. Using the third equation in (\( P \)), one projects the non-equilibrated part of the stress, namely \( \sigma^*_{n+1} - \sigma_Y \), to the stress–strain curve as can be seen in Figure 4. To be precise, setting \( r = (\sigma^*_{n+1} - \sigma_Y) (\sigma^*_{n+1} - \sigma_n)^{-1} \), we obtain \( \sigma^* = \sigma_Y + r E_i \sigma^* \). Note that the stress difference \( \sigma_Y - \sigma_n \) (and the corresponding strain) still belongs to the elastic range and needs not to be corrected.

3. \( \sigma^*_{n+1} > \sigma_n \geq \sigma_Y \).
   In this case, \( \sigma_n \) is in the plastic range and \( \sigma^*_{n+1} \) remains plastic. As in the latter case,
the non-equilibrated part of the stress needs to be corrected as indicated by Figure 4, i.e.,
\[ \sigma^* = \sigma_n + E_r(\sigma^\sigma_{n+1} - \varepsilon^\sigma_n). \]

4. -\(\sigma_{Y_2} \leq \sigma^*_{n+1} \leq \sigma_n, \sigma_n > 0:\)
In this case, unloading is performed and we recenter (or stay in) the elastic regime. Hence, we have to check that \(\sigma^*_{n+1}\) is larger than the compression yielding stress (recall, that \(\sigma_{Y_2} > 0\) and no correction is made, \(\sigma^* = \sigma^*_{n+1}\).

3.2 Space discretization: Wavelet spaces
Now, we are going to describe the use of wavelet bases for discretizing Problem 1 in space. To this end, we will give a very brief survey on wavelet methods. However, we restrict ourselves to those facts only, that will be used in the sequel and refer e.g., to [7, 14, 18] for textbooks and general surveys on wavelets.

A system of compactly supported \(L_2(0,1)\)-functions \(\Psi := \{\psi_\lambda : \lambda \in \mathcal{J}\}\) is called a wavelet basis if \(\Psi\) spans \(L_2(0,1)\) and the following norm equivalence holds
\[ \left\| \sum_{\lambda \in \mathcal{J}} d_{\lambda} \psi_\lambda \right\|_{L_2(0,1)} \sim \left( \sum_{\lambda \in \mathcal{J}} \left| d_{\lambda} \right|^2 \right)^{1/2}, \] (17)
where \(A \sim B\) denotes \(cA \leq B \leq CA\) for absolute constants \(0 < c \leq C\), and \(\mathcal{J}\) is an (infinite) set of indices. To be more precise, one can think of \(\lambda \in \mathcal{J}\) as a pair
\[ \lambda = (j, k), \quad |\lambda| := j, \]
where \(j \in \mathbb{Z}\) denotes the scale or level of \(\psi_\lambda\) whereas \(k\) indicates its location in space (e.g., the center of the support of \(\psi_\lambda\)). Equation (17) implies that \(\Psi\) is a Riesz basis of \(L_2(0,1)\). By Riesz Representation Theorem, (17) implies the existence of a dual wavelet basis \(\Psi' := \{\tilde{\psi}_\lambda : \lambda \in \mathcal{J}'\}\), i.e.,
\[ (\psi_\lambda, \tilde{\psi}_{\lambda'}\big)_{L_2(0,1)} = \delta_{\lambda, \lambda'}, \quad \lambda, \lambda' \in \mathcal{J}. \] (18)
The pair \(\Psi, \Psi'\) is often termed biorthogonal wavelet system.

The corresponding wavelet spaces which will be used to discretize Problem 1 are then given by suitable subsets of \(\Psi\) in the following way. For any (finite) subset \(\Lambda \subset \mathcal{J}\), we define \(\psi_\Lambda := \)
\{ \psi_{\lambda} : \lambda \in \Lambda \} \) and the induced wavelet space by

\[ S_\Lambda := \text{span} \Psi_\Lambda. \]

In the following, we consider the spaces \( S_j := S_{\mathcal{J}_j} \), where \( \mathcal{J}_j := \{ \lambda \in \mathcal{J} : |\lambda| \leq j \} \). These spaces are also often termed as multiresolution spaces.

It has been proven [14] that under suitable assumptions on the order of approximation and the regularity (direct and inverse estimates) of both \( \Psi \) and \( \hat{\Psi} \), (17) indeed holds for a whole range of Sobolev spaces including \( L_2(0, 1) \). To be more precise, one has

\[ \left\| \sum_{\lambda \in \mathcal{J}} d_\lambda \psi_{\lambda} \right\|_{H^s(0, 1)} \sim \left( \sum_{\lambda \in \mathcal{J}} 2^{2s|\lambda|} |d_\lambda|^2 \right)^{1/2}, \quad s \in (-\gamma, \gamma), \]  

(19)

where \( \gamma, \tilde{\gamma} > 0 \) depend on \( \Psi \), \( \hat{\Psi} \) and \( H^s(0, 1) \) denotes the usual Sobolev spaces [1]. It turns out that (19) is indeed the key for the strong analytical properties of wavelets for multilevel preconditioning and adaptivity [14].

Usually, the starting point for the construction of wavelet bases is a second system of functions \( \Phi_j := \{ \varphi_{j,k} : k \in I_j \} \) (where again \( I_j \) is a suitable set of indices), that are refinable, i.e.,

\[ \varphi_{j,k} = \sum_{\ell \in I_{j+1}} a_{j,\ell}^k \varphi_{j+1, \ell}, \quad k \in I_j, \]  

(20)

for suitable refinement coefficients \( a_{j,\ell}^k \in \mathbb{R} \). This implies that the generated spaces \( S_j = \text{span} \Phi_j \) are nested: \( S_j \subset S_{j+1} \). A similar construction is done for the dual system, which we will always indicate by a superscript ‘*’. The systems \( \Phi_j \), \( \hat{\Phi}_j \) are often referred to as primal and dual scaling systems.

The biorthogonal wavelet spaces \( W_j, \hat{W}_j \) are defined by

\[ W_j := S_{j+1} \ominus S_j, \quad \hat{W}_j := \hat{S}_{j+1} \ominus \hat{S}_j, \quad S_j \perp \hat{S}_j, \quad \hat{S}_j \perp W_j, \]

where orthogonality is to be understood w.r.t. the \( L_2(0, 1) \)-inner product. Finally, the wavelets \( \psi_{\lambda} = \psi_{j,k}, \psi_{\lambda} = \psi_{j,k} \) are constructed as a suitable basis for \( W_j, \hat{W}_j \), respectively, i.e.,

\[ W_j = \text{span} \Psi_j, \quad \hat{W}_j = \text{span} \hat{\Psi}_j. \]

(\( s_j, \hat{s}_j \)) is the wavelet basis constructed from a single mother wavelet \( \psi \) by taking integer translations of dyadic scaled versions of \( \psi \) [18], i.e.,

\[ \psi_{j,k}(x) := 2^{j/2} \psi(2^j x - k), \quad j, k \in \mathbb{Z}. \]  

(21)

On \([0, 1]\), one takes as many functions of the form (21) as possible (namely those that are supported strictly inside \([0, 1]\)) and perform suitable adaptions near the boundary points in order to preserve biorthogonality, regularity and approximation order, see e.g. [2, 10, 16, 20], where this list is far from being complete.

Nowadays, many families of wavelet constructions are available. We chose here biorthogonal spline wavelets, [9]. In this case, the primal scaling system is build by cardinal B-splines \( d \mathcal{N} \) of order \( d \in \mathbb{N} \). Then, dual scaling systems generated by some functions \( \delta \mathcal{N} \) for \( d + \hat{d} \) even can be found in the literature, [9]. The regularity and approximation order of \( \delta \mathcal{N} \) is related to \( \hat{d} \) and can be chosen arbitrarily high (with the dispense of increasing size of their support, however). The adaptation of this construction from \( L_2(\mathbb{R}) \) to \( L_2(0, 1) \) has been studied in many papers [2, 16, 20] and we choose the bases from [16] here.
3.3 Discretization in time: Newmark scheme In this subsection, we briefly review the definition and the main properties of the Newmark scheme which we use for the time discretization. Let us mention first, that this scheme is widely used for resolving evolutionary problems in elastic wave propagation and structural mechanics.

We consider a system of differential equations of the type

\[ M \ddot{X}(t) + K X(t) = F(t), \]  

where the matrices \( M \) and \( K \) are symmetric, \( K \) is positive (semi-)definite and \( M \) is positive definite. Then, for a given time step \( \Delta t = t_{n+1} - t_n > 0 \), the quantities \( X(t_n) \), \( X(t_{n+1}) \) and \( \dot{X}(t_n) \) are approximated by \( X^n \), \( X^{n+1} \) and \( \dot{X}^n \), respectively, due to the following relations

\[
M \ddot{X}^{n+1} + K X^{n+1} = F(t_{n+1}),
\]

\[
\dot{X}^{n+1} = \dot{X}^n + \Delta t [ (1 - \delta) \ddot{X}^n + \delta \ddot{X}^{n+1}],
\]

\[
X^{n+1} = X^n + \Delta t \dot{X}^n + \Delta t^2 \left[ (\frac{1}{2} - \theta) \ddot{X}^n + \theta \ddot{X}^{n+1} \right],
\]

resulting in the three-level scheme

\[
\frac{1}{\Delta t^2} M (X^{n+1} - 2X^n + X^{n-1}) + K \left[ \theta X^{n+1} + (\frac{1}{2} + \delta - 2\theta) X^n + (\frac{1}{2} + \theta - \delta) X^{n-1} \right] = F^*,
\]

with

\[
F^* := \theta F^{n+1} + (\frac{1}{2} + \delta - 2\theta) F^n + (\frac{1}{2} + \theta - \delta) F^{n-1},
\]

where the real parameters \( \theta \) and \( \delta \) have to be chosen in order to guarantee consistency and stability, [19]. Frequently used parameters are \( \delta = \frac{1}{2} \) and \( \theta \geq \frac{1}{4} \) which results in a second order scheme which is stable for all \( \Delta t \), [19].

3.4 Finite dimensional problems Now, we collect all the above described pieces and end up with a discretization of Problem 1.

In each time \( t_n \) we will approximate the solution by means of a wavelet expansion of the type

\[ u(t_n, x) \sim \sum_{\lambda \in \mathcal{J}_l} u_{\lambda}(t) \Psi_{\lambda}(x) := u_n(x), \]

where \( u_{\lambda}(t) \) are suitable coefficients. Then, the Galerkin approximation of (E) reads

\[
\sum_{\lambda \in \mathcal{J}_l} \left\{ \bar{u}_{\lambda}(t) (\rho \psi_{\lambda}, \psi_{\mu} u) + u_{\lambda}(t) (EA \psi'_{\lambda}, \psi'_{\mu} u) \right\} = (f, \psi_{\mu})_0, \quad \mu \in \mathcal{J}_l,
\]

where \((\cdot, \cdot)\) is the usual \( L^2 \) inner product. Hence, we obtain the system (22) with the following parameters

\[
M := M^\rho := \left( (\rho \psi_{\lambda}, \psi_{\mu})_0 \right)_{\lambda, \mu \in \mathcal{J}_l},
\]

\[
K := K^{EA} := \left( (EA \psi'_{\lambda}, \psi'_{\mu})_0 \right)_{\lambda, \mu \in \mathcal{J}_l},
\]

\[
X(t) = U_{\mu}(t) := (u_{\lambda}(t))_{\lambda \in \mathcal{J}_l},
\]

\[
F(t) = F_{\mu}(t) := ((f(t, \cdot), \psi_{\mu}(\cdot))_0)_{\mu \in \mathcal{J}_l}.
\]

This means that in each step of the Newmark scheme, we have to solve the linear system

\[
\tilde{A}^{n+1} \tilde{X}^{n+1} = \tilde{F}^{n+1},
\]

where

\[
\tilde{A}^{n+1} := \frac{1}{\Delta t^2} M^\rho + \theta K^{EA},
\]

\[
\tilde{X}^{n+1} = U_{n+1}(t_{n+1}),
\]

\[
\tilde{F}^{n+1} = \left( \frac{1}{\Delta t^2} M^\rho - \left( \frac{1}{2} + \delta - 2\theta \right) K^{EA} \right) X^n - \left( \frac{1}{\Delta t^2} M^\rho - \left( \frac{1}{2} + \theta - \delta \right) K^{EA} \right) X^{n-1} - F^*,
\]

where \( F^* \) is determined by (27).
3.5 Stress correction and wavelet bases In a certain sense, the philosophy of a Galerkin method for the discretization in space contradicts the elastic–predictor plastic–corrector strategy. The latter involves point values of the basis functions whereas the Galerkin discretization only uses integrals over (products of derivatives of) these functions. The reasons for using a Galerkin approach have been detailed above. However, the pointwise correction is physically adequate. Hence, we performed a mixture of the two methods as described below.

For the discretization in space we use a Galerkin method with biorthogonal spline wavelets. When it comes to the stress correction, we perform a change of basis to an interpolatory wavelet basis. The correction can easily be performed with the functions and also the change of basis is not very costly. The resulting unequilibrated force serves as a right–hand side in an elliptic problem. Hence, it is rather easy to perform an $L_2$–projection of this function in order to determine the entries of the right hand side. This approach allows us to combine two advantages, namely the use of mathematically founded Wavelet–Galerkin methods with biorthogonal wavelets and an efficient stress correction using interpolatory wavelets. A numerical analysis of the best choices for the coupling between Galerkin approach and pointwise correction is currently under investigation.

4 Numerical Results

In this section, we present some of our numerical experiments. Let us mention that the code has been written in C++ using the MultiLevel Library [5]. The code has been validated with the code from [11] and also by comparisons with standard Finite Element code FEAP described in [27].

The input data for our test example have been chosen as:

\[ T = 0.4 \quad \Delta t = 0.01 \quad \sigma_{Y_1} = \sigma_{Y_2} = 1600 \]

\[ A(x) \equiv 100 \quad \rho(x) \equiv 7.85 \quad E = 2100000 \]

\[ f(t, x) := \chi_{[0.0, 0.7]}(x) a \sin(\omega t) \quad a = 8 \cdot 10^3, \omega = \frac{20}{\pi} \]

and within the space domain $x \in [0, 1]$. The solution is shown in Figure 5, where on the left we have displayed the displacement and on the right the characteristic stress/strain curve at the two points $x = 0$ and $x = 0.98$. This example is particularly interesting since all hardening cases described in Figure 4 in fact appear. Moreover, in Figure 6, we have plotted the resulting plastic indicators. We obtain two plastic waves starting from the boundaries.

![Displacement](image)

![Strain/stress curve](image)

Figure 5: Displacement (left) and characteristic strain/stress curve (right).

Our first experiment concerns discretizations of different orders, i.e., biorthogonal B–spline wavelets for different choices of the parameter $d$ as described at the end of Section 3.2. Of course, the solution quantitatively is the same as the one indicated by the Figures 5 and 6. Here, we
want to study some qualitative aspects. In Figure 7, we show data for the solutions concerning $d = 2, 3, 4$ all for level $J = 10$ using piecewise linear interpolatory wavelets up to level $J_{\text{red}} = 10$ for the stress correction. The first picture in Figure 7 shows the error for $d = 2$. Note that the error is in the range of $10^{-7}$. The two remaining pictures in Figure 7 show the difference between the solutions for $d = 2, d = 3$ (in the middle) and $d = 2, d = 4$ (right). We see that the distance is almost negligible except two very localized regions near $x = 0$ and $x = 0.7$. It can be seen from Figure 6 that these are exactly those positions of the plastic wave. It seems that high

![Graph](image)

Figure 6: Plastic indicators. The $x$-axis corresponds to the time steps $(n = 1, \ldots, 40)$ and the $y$-axis to the points $i/1024, i = 1, \ldots, 1024$ in space.

![Graph](image)

Figure 7: Error for $d = 2$ (left) and difference of the solutions for $d = 2, 3$ (middle) and $d = 2, 4$ (right).

order discretizations do not pay off for this example even if they again permit to detect regions where plasticity is spreading. In order to understand this behaviour, we consider the parts of the solution corresponding to different wavelet levels. In Figure 8, these parts are displayed for $d = 2$. We see that the details (i.e., the wavelet parts) in fact reflect the beginning of a plastic wave.

We now come to the relationship between the wavelet coefficients and plastic phenomena that should justify the use of more advanced wavelet methods. Therefore, in Figures 9 and 10, we show the scaling and wavelet coefficients for $d = 2$ and $d = 3$, respectively. Obviously, the behaviour of the coefficients for $d = 3$ reflects the occurrence of plasticity much better. We see large peaks at the borders of the waves. These pictures show in particular that the loss of high order convergence is (at least partially) due to the interplay of the different wavelet bases in the stress correction. Since the support of the trial functions for $d = 3$ and $d = 4$ is larger than the one of the piecewise linear interpolatory wavelets, the $L_2$-projection of the unequilibrated force may cause an error.

Note that we keep using piecewise linear interpolatory wavelets for the stress correction.
This could limit the maximal reachable order, even though the stress correction only effects the right-hand side. Thus, one would have to increase either the level \( J_{\text{max}} \) or the order of the interpolatory wavelets. Since these effects are highly localized for the high order system, we propose an adaptive strategy which will be considered in a forthcoming paper. Finally, the nature of the problem itself might also limit the order of approximation. In fact, the regularity theory of such problems is still a somewhat open field, [22].

In the light of the above result, we confine the next discussions to piecewise linear trial and test functions for the Wavelet-Galerkin method. First of all, we display the error in the Sobolev norms \( \| u(t, \cdot) - u_j(t, \cdot) \|_{H^s([0,1])}, s = 0, 1 \) at some time \( t \in [0, T] \) for different values of \( j \). Here, \( u_j \) denotes the computed Galerkin solution w.r.t. \( S_j \) and \( u \) is the 'exact' solution. The latter one is computed by using trial and test spaces \( S_j \) with a large value for \( j \).

In Figure 11, we have displayed the evolution of these errors in time. We see that the comparably big values of the errors correspond to the occurrence of the plastic waves.

Now, we consider two choices for \( t \) and monitor the errors in dependence of the maximum level. Our first choice is the final time \( t = 0.39 \). The two pictures in Figure 12 show the errors in semilogarithmic scale (left) and the rate of convergence (right). We obtain almost the same rate of convergence as for the Wavelet-Galerkin Method applied to an elliptic problem. The same information as in Figure 12 for \( t = 0.39 \) is displayed in Figure 13 for \( t = 0.16 \). Note that at this time, no plastic wave occurs in the displacement, see Figure 6, and the general behaviour is similar as in the above case.

In the next experiments, we fix the highest level for the Galerkin discretization to \( J = 9 \) and consider the \( L_2 \)-error for different choices of \( J_{\text{max}} \), i.e., the maximum level for the stress correction. In Figure 14, the corresponding error is displayed for \( d = 2 \) and the choices \( J_{\text{max}} = 3, 4, 5, 6 \). We obtain a smoother and smoother error for increasing values of \( J_{\text{max}} \). In the Figures 15 and 16, we display the \( L_2 \)-error and the rate of convergence for \( t = 0.16 \) and \( t = 0.39 \). We cannot detect an obvious rate of convergence from these pictures. However, the interplay between the wavelet bases for the Galerkin method and the stress correction seems to be of great importance.

5 Conclusion

We have introduced an elastic predictor/plastic corrector method for elastoplastic problems using biorthogonal wavelets for the Galerkin method for elasticity and interpolatory wavelets for the plastic corrector step.

Our numerical results show that the wavelet coefficients of high order systems are local indicators for the occurrence of plastic waves. These waves have a great influence on the rate of convergence of the whole method.

The appropriate choice of the wavelet systems for the Galerkin method and the stress correction is of great importance. Adaptive wavelet methods with local refinement at the boundaries of plastic waves offer great potential to reduce the number of unknowns as well as the computational cost.

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Figure 8: Scaling and wavelet parts of the computed solution for the different levels, $d = 2$. 
Figure 9: Scaling and wavelet coefficients of the computed solution for the different levels, $d = 2$. 
Figure 10: Scaling and wavelet coefficients of the computed solution for the different levels, $d = 3$. 
Figure 11: $L_2$- and $H^1$-error over $t$ for different maximal levels $J$.

Figure 12: $L_2$- and $H^1$-error over the maximal level for $t = 0.39$.

Figure 13: $L_2$- and $H^1$-error over the maximal level for $t = 0.16$. 

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Figure 14: Error for $J_{\text{Res}} = 3, 4, 5, 6, d = 2$.

Figure 15: $L_2$-error (left) and rate of convergence (right) w.r.t. $J_{\text{Res}}$ at $t = 0.16$. 
Figure 16: $L_2$-error (left) and rate of convergence (right) w.r.t. $J_{RMS}$ at $t = 0.39$. 